

## PREFACE

This book grew out of lectures I gave at the University of California (Berkeley) in the spring semester of 1964. The lectures were planned to cover a specific and beautiful topic in riemannian geometry, the geometry of spaces of constant curvature, presenting known progress toward the solution of its outstanding problems. Halfway through the semester I was trapped by the subject and lingered on to solve a few of those problems. The state of the theory now warrants a book, and I hope that this is the book so warranted.

The theory of spaces of constant curvature might be said to have originated with euclidean geometry. But it really began with the Gauss-Lobatchevsky-Bolyai inventions of a non-euclidean geometry in the early nineteenth century. That geometry is now called the synthetic geometry of the hyperbolic plane. Its discovery marked the end of attempts to prove Euclid's parallel postulate from the other postulates of euclidean geometry, for it has the property that (infinitely) many parallels to a given line pass through any point off that line. In 1854 Riemann invented two non-euclidean geometries with the property that two distinct lines cannot be parallel. Those geometries are now known as the synthetic plane spherical and elliptic geometries. At the same time Riemann laid the foundations for riemannian geometry and exhibited riemannian metrics of arbitrary given constant curvature. Then, in 1868, Beltrami proved the consistency of the hyperbolic and spherical geometries (relative to euclidean solid geometry) by realizing them as the intrinsic geometries of well-known surfaces in euclidean space. Those surfaces are the pseudosphere for hyperbolic geometry, which has constant negative curvature, and the ordinary sphere for spherical geometry, which has constant positive curvature. In addition to causing general acceptance of the non-euclidean geometries, Beltrami's proof moved them into the domain of the then new riemannian geometry. Within a dozen years there was considerable interest in surfaces of constant curvature. In 1891, Killing published a book in which riemannian manifolds of arbitrary dimension and arbitrary constant curvature were exhibited and in which the problem of finding all riemannian manifolds of constant curvature was well formulated. That problem received additional impetus from Einstein's invention of general relativity and was put in proper perspective by É. Cartan's development of the theory of symmetric spaces.

The purpose of this book is to describe the classification problems in the theory of spaces of constant curvature and the theory of symmetric spaces. Results to date are given and a number of additional problems

are solved. The coverage is best explained by the following description of the contents of this book and by reference to the table of contents.

The first half of the book (Parts I, II, and III) is concerned with spaces of constant curvature *per se*. The reader is expected to have some familiarity with advanced calculus, point set topology, linear algebra, and elementary group theory.

PART I (riemannian geometry) consists of two chapters. Chapter 1 develops the concepts of differentiable manifold and linear connection, digresses for an exposition of the theory of covering spaces, and concludes with a treatment of global affine differential geometry. The results are essentially standard but many proofs are new. Chapter 2 develops the Levi-Civita connection and the concept of sectional (riemannian) curvature, and touches on the geometry of spaces of constant curvature. Then it illustrates the two main techniques of this book, the first by the isometric classification of riemannian 2-manifolds of constant curvature  $K \geq 0$ , and the second by the isometric classification of riemannian homogeneous manifolds of arbitrary constant curvature. These classifications are known but not standard.

PART II (the euclidean space form problem) consists of one chapter. Chapter 3 describes the present state of the theory of flat (zero curvature) complete riemannian manifolds by presenting the Bieberbach Theorems, applying them to the euclidean space form problem, and using the resulting structure theory to obtain the isometric classification of flat complete riemannian 3-manifolds. That classification is new, although it is only a refinement of other types of classifications of Nowacki and Hantzsche-Wendt. The chapter ends by considering some lines of research that look hopeful for the future.

PART III (the spherical space form problem) consists of four chapters. It gives the isometric classification of complete riemannian manifolds of constant positive curvature, solving the "Clifford-Klein spherical space form problem" proposed by Killing in 1891. That solution is new, and it forms the nucleus of this book. Chapter 4 is preparatory, developing the representation theory of finite groups from the viewpoint of Frobenius' reciprocity. Chapter 5 develops elementary  $p$ -group theory and then applies it with the representation theory to obtain Vincent's partial solution to the spherical space form problem. Chapter 6 is the classification of the family of finite groups which occurs in the spherical space form problem. Chapter 7 is the synthesis, resulting in the solution.

The second half of this book (Parts IV and V) deals with various natural extensions of the class of spaces of constant curvature. The pace is faster than in the first half, and the reader is expected to know the basic facts on compact topological groups, Lie groups, and Lie algebras.

PART IV (space form problems on symmetric spaces) consists of three chapters. It is concerned with the problem of extending the solution of the spherical space form problem to riemannian symmetric spaces of nonnegative curvature. Chapter 8 is a fairly complete introduction to riemannian symmetric spaces and two point homogeneous spaces. It contains the classification, including the linear isotropy representations, and a fair amount of new material toward the end. Chapter 9 extends the solution of the spherical space form problem to a large class of compact symmetric spaces, and Chapter 10 deals with symmetric spaces of non-negative curvature.

PART V (space form problems on indefinite metric manifolds) consists of two chapters. Chapter 11 is concerned with constant curvature indefinite metric manifolds, while Chapter 12 treats the generalization to indefinite metric of the two-point-homogeneous riemannian manifolds.

The working method in this book is a mixture of E. Cartan's method of moving frames and the theory of groups. Here I gratefully acknowledge my debt to my teacher S.-S. Chern, who stimulated and guided my interest in differential geometry. Thanks for guidance are also due A. A. Albert for teaching me how to sit down with an algebraic problem.

This book was originally written as a monograph but in part due to the selection of material has been used as a text. A number of people made useful suggestions which are incorporated into this edition, notably B. O'Neill, V. Ozols, R. E. Stong, W. Boothby and Wu-yi Hsiang. In contrast to the original 1967 edition, Chapter 1, Chapter 2 and the first half of Chapter 8 now form a reasonable introduction to differential geometry and symmetric spaces.

Special thanks are due to my wife Lois for her continuous encouragement and cooperation while I was writing and preparing revisions.

*Berkeley, June 1977*

*J. A. W.*

## PREFACE TO THE SIXTH EDITION

Since publication of the fifth (1984) edition of this book there has been a tremendous amount of activity in discrete subgroups of Lie groups and algebraic groups. This activity had emphasis in several areas, especially differential geometry, harmonic analysis, algebraic geometry and number theory. It also had applications via Fourier transform theory to signal processing and other areas.

Most of the new material in this sixth (2010) edition represents an attempt to indicate some of these developments. Much of this is done in Chapter 3 and in the Appendix to Chapter 12. Chapter 3 has some new results and an indication of updates in the section on flat homogeneous pseudo-riemannian manifolds. The Appendix to Chapter 12 sketches some background and a brief description (sometimes just consisting of current references) of the more recent work on discrete subgroups of real Lie groups. There the emphasis is on application to pseudo-riemannian geometry and pseudo-riemannian quotient manifolds, including of course the riemannian case. There has also been an enormous amount of work on spaces of functions on those quotients, but that is well beyond the scope of this book.

I thank Oliver Baues for his generous advice and updates concerning the revision of Chapter 3. Thanks also to Jonathan Wahl in connection with the change in the Remark on page 170. I was tempted to modernize the finite group theory in Chapter 6, but that could have made it inaccessible to many differential geometers, and I thank James Milgram and C. T. C. (Terry) Wall for convincing me not to do it. Finally, my thanks to Hillel Furstenberg, David Kazhdan and Toshiyuki Kobayashi for updates and references in the Appendix to Chapter 12.

In this new material, note that citations not in the 1986 “References” section are in “Additional References” just after.

As ever, special thanks are due to my wife Lois for her support while I was preparing this new edition.

*Berkeley, July 2010*

*J. A. W.*