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# Research in Collegiate Mathematics Education. IV

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## Contents

<b>Preface</b>	vii
<i>Ed Dubinsky, Alan Schoenfeld, Jim Kaput, Cathy Kessel</i>	
<b>Teaching and Learning Calculus: What Can Be Learned from Education Research and Curricular Changes in France?</b>	1
<i>Michèle Artigue</i>	
<b>Evaluating Calculus Reform: A Review and a Longitudinal Study</b>	16
<i>Betsy Darken, Robert Wynegar, and Stephen Kuhn</i>	
<b>The Need for Evaluation in the Calculus Reform Movement: A Comparison of Two Calculus Teaching Methods</b>	42
<i>Susan L. Ganter and Michael R. Jiroutek</i>	
<b>A Longitudinal Study of the C<sup>4</sup>L Calculus Reform Program: Comparisons of C<sup>4</sup>L and Traditional Students</b>	63
<i>Keith E. Schwingendorf, George P. McCabe, and Jonathan Kuhn</i>	
<b>Understanding Sequences: A Tale of Two Objects</b>	77
<i>Michael A. McDonald, David M. Mathews, and Kevin H. Strobel</i>	
<b>A Theoretical Framework for Analyzing Student Understanding of the Concept of Derivative</b>	103
<i>Michelle J. Zandieh</i>	

<b>Why Can't Calculus Students Access Their Knowledge to Solve Non-routine Problems?</b>	128
<i>Annie Selden, John Selden, Shandy Hawk, and Alice Mason</i>	
<b>Lasting Effects of the Integrated Use of Graphing Technologies in Precalculus Mathematics</b>	154
<i>William O. Martin</i>	
<b>Visual Confusion in Permutation Representations</b>	188
<i>John Hannah</i>	
<b>Factors, Divisors, and Multiples: Exploring the Web of Students' Connections</b>	210
<i>Rina Zazkis</i>	
<b>On Student Understanding of AE and EA Quantification</b>	239
<i>Ed Dubinsky and Olga Yiparaki</i>	

## Preface

Welcome to the fourth volume of *Research in Collegiate Mathematics Education (RCME IV)*. Student learning and calculus are major themes in this volume. As in previous volumes, these are examined in a variety of ways. Seven of the eleven articles that comprise *RCME IV* concern different aspects of calculus. The first two give overviews of calculus reform in France and in the United States. The next two are small- and large-scale longitudinal comparisons of students who were enrolled in first-year reform and traditional courses. Four detailed studies of students' understandings of calculus and related topics follow. We then switch gears to focus more directly on relationships between instruction and students' understandings for courses other than calculus—abstract algebra and number theory—and finish with a cross-sectional study of a cross-cutting concept—quantifiers.

### Calculus reform

Michèle Artigue gives an overview of calculus in France, including relevant research as well as a brief history of curricular change. Unlike the United States, France has a national program of study; teaching calculus at the high school level became widespread at the beginning of the 20<sup>th</sup> century. Alongside the national curriculum has been a coherent research enterprise which, for some decades, has explored mathematics teaching and learning at the upper high school (including calculus) and college levels. This research enterprise, much more homogeneous than in the U.S., is grounded in a particular sociocultural perspective, including the idea (called the “didactic contract”) that students and teachers enter the classroom with a set of mutual, though often tacit, expectations, which play a strong role in shaping their classroom behavior. French cognitive studies have also focused on “epistemological obstacles,” conceptual issues that have proven difficult both historically for the field and for individual students when learning the material. Artigue describes some of the theoretical frameworks used in research and some of the different kinds of student difficulties relevant to calculus which have been documented by research. This is followed by an account of the evolution of the teaching of calculus at the lycée level (grades 10–12). The syllabus changed in the 1960s and 1970s due to the influence of the Bourbaki. Another change occurred in 1982, this time influenced by the findings of mathematics education research, and the curriculum focused on approaches that were more intuitive than the formal approaches inspired by the Bourbaki. The situation in France may be of particular interest to readers from the United States because about 70% of French students take at least two years of calculus in high school, hence the French have been concerned far longer with the problem of how to make calculus accessible to the majority of students. (The following statistics give a sense of how many high school students in the United

States take calculus. In 1995 about 4% of grade 12 students took an Advanced Placement calculus exam. The National Science Foundation, extrapolating from a sample, indicates that about 11% of grade 12 students took a calculus course in 1993.) The French approach focusing on variation and approximation has had its successes, but some problems remain and new problems have arisen.

In the United States, calculus reform might be said to have begun with the Tulane conference of 1986. There were various reasons for concern. Calculus courses had high drop-out rates; the content of the courses wasn't adequate for further study of mathematics, science, and engineering; students didn't seem to understand that content well; and, unlike courses of the past, 1980s calculus courses did not appear to inspire many students to become mathematicians. After more than a decade, it is natural to ask whether reform has successfully addressed these concerns. However, any answer must take into account the different instantiations of calculus reform. They may include the use of technology, small groups during class, writing, or structured small-group sessions that supplement classes run in a traditional manner. Different textbooks and curriculum programs include these elements in different ways, along with treatments of topics that break with tradition. Betsy Darken, Robert Wynegar, and Stephen Kuhn give an overview of these different kinds of calculus reform and summarize the research on the effects of different elements of reform as well as the effects of different reform texts and programs. With the caveat that research on reform is limited, they conclude that calculus reform is doing no harm and may even be doing some good.

Darken and her colleagues contribute to the research on reform with a longitudinal comparison of students from first-year courses using a reform text (Ostebee and Zorn) and a traditional text. Students' course grades were compared for reform and traditional first-year calculus—and for “unreformed” second-year courses (Differential Equations and Multivariable Calculus). Students in the reformed first-year course withdrew less often although the grades for reform and traditional courses were not significantly different. Consistent with this, success rates (passing grades divided by all grades including withdrawals) were significantly different for the two courses. Student performance and retention were similar in the second-year courses.

Susan Ganter and Michael Jiroutek studied a different instantiation of calculus reform, a course that was traditional except for the addition of projects (written up every four weeks) and the replacement of one of four weekly class meetings by a computer lab. This reflects a conceptualization of reform as a change in teaching methods rather than a change in content or the way in which it is organized. Ganter and Jiroutek focus on two questions: Does the change in “delivery method” affect students' mastery of “basic” calculus skills—taking derivatives, finding equations of tangent lines, and the like? The answer in this case was yes, students in reform courses performed more poorly on tests of these skills than did their counterparts in traditional courses. Like Darken and her colleagues, Ganter and Jiroutek examined grades in later courses and found no significant differences for students from reform and traditional courses. Unlike the grades analyzed by Darken et al., these included grades in science as well as mathematics courses.

Calculus, Concepts, Computers, and Cooperative Learning (C<sup>4</sup>L) is a reform calculus program that involves technology and has been extensively studied. Small-scale in-depth comparisons of the understandings of students taking C<sup>4</sup>L and traditional courses have been made. But what about long-range issues like students'

grades in later courses—or whether they take those later courses? Keith Schwingendorf, George McCabe, and Jonathan Kuhn address these questions with a large-scale study of the mathematics grades of 4636 Purdue University students who enrolled in first-year calculus in 1989, 1990, or 1991. Treatment A versus treatment B comparisons of such data often face methodological difficulties: in general—and in this case—students are not randomly assigned to courses. Schwingendorf et al. note that such difficulties may often be inevitable—hence methods for comparing calculus programs when students are not randomly assigned are needed. Their analysis controlled for factors that appeared to be important: predicted grade point average (a combination of entrance exam scores and high school grades), major, and gender. On average, C<sup>4</sup>L students earned higher grades in calculus courses, took more courses beyond calculus and had slightly better grades in those courses.

### Student understandings

What do students who succeed in calculus know? The next four articles give detailed accounts of students' understandings and abilities. Phenomena common to all four articles are: What each student knows or can do may differ considerably, even in a group of students receiving As or Bs. And such students may display surprising gaps in their knowledge.

The first two articles concern students' understandings of two important concepts in calculus: sequence and derivative. Michael McDonald, David Mathews, and Kevin Strobel use the Action-Process-Object-Schema (APOS) framework to analyze students' understanding of sequences. According to this framework, sequences can be understood as a process—a list of numbers, or as an object—in brief, a function whose domain is the integers. McDonald and his co-authors interviewed 21 students who would be considered successful according to standard criteria—with two exceptions they had earned As or Bs in second-semester calculus. One group of students had taken a traditional calculus course and the other a C<sup>4</sup>L course. These students' interviews and written work suggest that the C<sup>4</sup>L students are more likely to understand sequences as functions than traditionally taught students.

Michelle Zandieh describes a framework for analyzing student understanding of derivative and illustrates its use with a case study of nine high school students. Her framework is related to the PO of APOS: it consists of three layers of pairs where the first element of the pair is a process and the second is the object that is the result of that process. The first layer concerns ratio, the second, the limit of that ratio (derivative at a point), and the third, the collection of those limits (the derivative function). Each element of these pairs can be interpreted in different contexts: symbolically, graphically, in terms of rate (which leads to particular symbolic forms), and in terms of velocity (which leads to connections with physical experiences as well as with metaphorical uses of “velocity,” “increase,” and so on). Such a framework suggests just how complicated an adequate understanding of derivative might be. Zandieh uses her framework to analyze interviews of nine high school students enrolled in an Advanced Placement calculus course. Although these students attended the same class, their initial understandings of derivative fell in quite different categories of the framework. As the course progressed and students' understandings became more complete, they became more similar. The

study suggests that: students' initial understandings of a particular topic may differ and students with similar initial understandings may not add to those partial understandings in the same ways.

Annie Selden, John Selden, Shandy Hauk, and Alice Mason provide data that remind us of some of the reasons for reform—specifically, that traditionally-taught students are not doing as well as we'd like. Selden et al.'s article is the third in a series of related studies. The previous studies analyzed traditionally-taught calculus students' responses to “moderately non-routine” calculus problems. The results were discouraging: Although responses on a test of related skills suggested that the students had the necessary skills, two-thirds of the students who earned As or Bs in calculus did not solve one problem and none of the students who earned Cs did. Do students develop the ability to solve such problems later in their mathematical lives? Perhaps. Selden et al. asked students who were finishing a traditional calculus/differential equations sequence to solve the same “moderately non-routine” calculus problems. Slightly more than half (16 of the 28 students) failed to solve any of the non-routine problems. Selden et al. analyze possible explanations. Was it absence of relevant knowledge? As in Selden et al.'s earlier studies the differential equations students were tested on the algebra and calculus necessary to solve the non-routine problems. Although results of the tests indicated that students were familiar with the calculus that could be used to solve the non-routine problems, solution methods tended to be algebraic. Selden et al. suggest that students may have the relevant knowledge but are not accessing it for “moderately non-routine” calculus problems—students do not have “problem situation images” that include “tentative solution starts,” in the authors' terms. They describe a potential remedy: courses that include experiences fostering students' constructions of rich problem situation images that stimulate their recall of relevant knowledge.

William Martin also examined the effects of a previous course on students. His analysis suggests pedagogical implications. The question was: What is the effect of a college algebra course that incorporates graphing calculators? Martin interviewed 18 students toward the end of their first semester of calculus. Nine of the students had taken a college algebra course that included the use of graphing calculators. However, the course was not much changed from the traditionally taught college algebra course at the same university taken by the nine other students that Martin interviewed. Instructors had not received special instruction in the use of graphing calculators nor were their courses dramatically different from the traditionally taught courses. The students' performance was reminiscent of Selden et al.'s differential equations students: They performed well on routine pre-calculus tasks and had little success on conceptual items involving calculus. Although Martin found few statistically significant differences, he noticed an interesting trend. Students in both groups used inappropriate strategies apparently cued by superficial features of the tasks, but those in the graphing calculator group did so less often. Martin hypothesizes that, when faced with a problem traditional students may be more inclined to try standard procedures and that graphing calculator students may be more inclined to focus on understanding the problem. As Selden et al. point out, novices in a particular domain tend to focus on surface characteristics of a task. Martin's study suggests that the instruction that the graphing calculator students received may have helped them to focus on the structure of a task rather than its surface features.

## Teaching

We now turn to studies concerning more detailed aspects of instructional design and their interaction with students' learning. John Hannah taught two instantiations of an abstract algebra course. His study focuses on his students' understanding of  $D_4$  (the dihedral group of all symmetries of a square). In *RCME II*, Zazkis and Dubinsky discussed students' encounters with two representations of elements of  $D_4$ . These representations correspond to different ways of viewing the symmetries of a square: globally as transformations (reflections and rotations of the square) or locally as permutations (changes in position of its vertices). Computations with either representation should give the same result, but most of the students interviewed by Zazkis and Dubinsky got different results for compositions of the same elements, depending on whether they used transformations or permutations. These were connected with differences in notation (does composition go from left to right or right to left?) and visualization (does one focus on the vertices of the square moving against a fixed background or on the background positions receiving different labels?). That's a problem. What can instruction do to address it? For the first instantiation of his course, Hannah designed a system of labels and diagrams: for example, the four vertices of a square were labeled and the "background" against which the square had four positions labeled—both with numbers from 1 to 4. Despite this, in computing compositions of elements of  $D_4$  using transformations and permutations, his students' responses were similar to those of Zazkis and Dubinsky. However, his students' suggestions that labels for the "background" be letters and those for vertices be numbers, led Hannah to a more successful refinement of the diagram-label system, which he used in the second instantiation of his course. But, although Hannah's students were more successful than those of the previous year in their computations with elements of  $D_4$ , he found that his students had difficulties in analogous situations: calculating compositions of transformations using three-dimensional models and calculating compositions of permutations using numerical arrays. Hannah's article suggests that instructors may find it helpful to attend to students' perceptions (which may be discovered from previous research and in more specific form from one's own students) and that consciousness of those perceptions may aid in redesigning instruction.

Rina Zazkis studied a very different group of students, prospective elementary teachers who were enrolled in her number theory course. She also used a different method, clinical interviews. She began her study with the intention of examining her students' understanding of three related terms (factor, multiple, and divisor), but found that connections with students' prior knowledge were impossible to ignore. Although the prospective teachers had studied factors and divisors for three weeks of Zazkis's course, in interviews they displayed incorrect understandings of these terms based on their pre-college schooling. This study suggests that teaching preservice teachers may be a very different enterprise than teaching children or mathematics majors, because prior knowledge may play a different role. As Zazkis puts it: "What often occurs in a content course for preservice elementary school teachers is not construction of new meanings or concepts, but reconstruction of previously constructed meanings." This poses a problem for research: How is learning different from re-learning? Can it be explained within existing theoretical

frameworks? And it poses a problem for instruction: Can an undergraduate program help prospective elementary teachers re-construct meanings for terms learned during six years of elementary school?

Like Zazkis’s preservice teachers, the science, mathematics, and mathematics education majors studied by Ed Dubinsky and Olga Yiparaki brought previous understandings to their interviews. Those interviews did not address a particular course topic, but rather a topic that occurs throughout mathematics—quantifiers, in particular, “there exists” and “for all,” and alternations of the two. Undergraduate mathematics majors do often not study quantifiers explicitly as a separate topic, but they encounter them throughout their studies. Dubinsky and Yiparaki addressed the question of how students’ interpretations of words used in everyday contexts, such as “all,” “every,” and “there is” might support their interpretations of similar terms used in mathematical statements. The findings were reminiscent of Zazkis’s—in trying to interpret everyday and mathematical statements, the students were inclined to draw on their experiences and use context rather than syntax. This inclination seems to explain why interpreting mathematical rather than everyday statements was far more difficult for the students—they had far less mathematical than everyday experience and their everyday experiences had not provided them with a strong sense of syntax. At the end of each interview, the students were asked to play a game involving alternations of quantifiers for three of the statements. In general, this seemed to help the students. Both the findings, that students tend to rely on context rather than syntax and that instruction concerning syntax aids student understanding, suggest that instructors should not rely exclusively on analogies with everyday experience in helping students learn to interpret statements involving quantifiers. Instead, instruction might create experiences that allow students to learn about syntax.

All told, these papers show that a growing community of researchers is beginning to systematically gather and distill data regarding collegiate mathematics teaching and learning. We look forward to more reports in future volumes.

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