

Preface

What happens when a student completes a course in “AP Calculus”, followed by a routine course in multivariable calculus, a computational course in linear algebra, and a formulaic presentation of differential equations? It is time for some real mathematics. There is still a world of interconnected labyrinths to explore, unscalable mountains to be climbed, and lands of mystery to be discovered. The first step in this is a rigorous course in one-variable analysis. This begins with a study of an ordered field in which the least upper bound property holds (not to be confused with a complete ordered field; see the project in Section 2.7.3). Here the student meets Bolzano-Weierstrass, Heine-Borel, and a rigorous treatment of one-variable differentiation and integration with careful attention paid to the pervasive presence of the Mean Value Theorem. Then what? That is exactly the reason behind this book.

Learning serious mathematics is about engaging with problems, from kindergarten to graduate school and beyond. The preliminaries for reading this book are already contained in the author’s book *Tools of the Trade* [27]. The first two chapters of *Tools* are Appendices A and B of this book. The reader who is familiar with that material can jump right into Chapter 1. The sequence of topics can be gleaned from the table of contents, so I will not dwell on that.

There are three features here that should be discussed explicitly, especially since they are important for the use of this book as a text. The first, and most important, is the collection of exercises. These are spread throughout the chapters and should be regarded as an essential component of the student’s learning. Some of these exercises comprise a routine follow-up to the material, while others will challenge the student’s understanding more deeply. The second feature is the set of independent projects presented at the end of each chapter. These projects supplement the content studied in their respective chapters. They can be used to expand the student’s knowledge and understanding or as an opportunity to conduct a seminar in Inquiry Based Learning (IBL) in which the students present the material to their class. A brief glance will show that the independent projects cover a wide range of interesting topics that hint at advanced areas of mathematics. The

third feature is the real kicker in this business. We list a series of challenge problems that increase in impossibility as the chapters progress.

I have taught the material in this book many times over the past forty-five years. The main audience has been the students in Honors Analysis (MATH 207-208-209) at the University of Chicago. These students are drawn from two sources. The first is the collection of sophomores who have excelled at Honors Calculus in their first year at Chicago. The second is a selection of pyrotechnically endowed freshmen who are capable of attacking mathematics at this level.

Some of the texts I have recommended during this time are T. Apostol, *Mathematical Analysis* [2], J. Dieudonné, *Foundations of Modern Analysis* [3], A. Kolmogorov and S. Fomin, *Introductory Real Analysis* [12], S. Lang, *Undergraduate Analysis* [13], L. Loomis and S. Sternberg, *Advanced Calculus* [18], W. Rudin, *Principles of Mathematical Analysis* [24], and, more recently, C. Pugh, *Real Mathematical Analysis* [22]. All of these books have some nice features. The intersection with the material of the present book is highly nontrivial. Nonetheless, I have always liked the idea of challenge problems, independent projects, and the organization of the mathematics presented here. For example, it is about time that mathematicians came to grips with Fourier analysis on p -adic fields, since it is an integral part of current-day research.

At the beginning of each chapter, I have included a quote from a well-known mathematician (or group of mathematicians) that gives a certain perspective on the material in that particular chapter. We leave it to the reader to speculate as to whether this perspective is that of the author. These quotes express a variety of opinions, and I have found them to be informative and sometimes amusing. The quote of A. Zygmund at the beginning of Chapter 7 is particularly relevant to the mathematics in the text.

FURTHER ADVICE TO THE STUDENT

(If you do not care about advice, just get started with the challenge problems in Chapter 1.)

It would be much better for both of us if I were sitting on a desk at the front of the class and talking to you. Nevertheless, a few words of warning are in order. First of all, you should scan the material in Appendices A and B and make sure you feel comfortable with it. Throughout the text, there are many references to these appendices. Secondly, if you find a particular exercise in the text to be quite simple and the next exercise to be very difficult, that's just the way it is. When doing mathematics, you never know when a road that seems smooth is going to have a pothole that is ten feet deep. Thirdly, if you take the challenge problems seriously, you will find that some of these problems can require looking somewhere other than Wikipedia. In that process, you can discover that there is lots of good stuff

in libraries. In any mathematics course of consequence, students should always be willing and ready to find other approaches to the proofs and solutions that are given “in class”. In most cases, after Chapter 1, the challenge problems are related to material that has been covered earlier in the text. Good luck and enjoy.

Paul J. Sally, Jr.
Chicago, Illinois
July 31, 2012