Preface

Basic complex variables is a very popular subject among mathematics faculty and students. There is one big theorem (the Cauchy Integral Theorem) with a somewhat difficult proof, but this is followed by a host of easy to prove consequences that are both surprising and powerful. Furthermore, for undergraduates, the subject is refreshingly new and different from the analysis courses which precede it.

Undergraduate complex variables is our favorite course to teach, and we teach it as often as we can. Over a period of years we developed notes for use in such a course. The course is a one-semester undergraduate course on complex variables at the University of Utah. It is designed for junior level students who have completed three semesters of calculus and have also had some linear algebra and at least one semester of foundations of analysis.

Over the past several years, these notes have been expanded and modified to serve other audiences as well. We have used the expanded notes to teach a course in applied complex variables for engineering students as well a course at the first year graduate level for mathematics graduate students. The topics covered, the emphasis given to topics, and the choice of exercises were different for each of these audiences.

When taught as a one-semester junior level course for mathematics students, the course is a transitional course between freshman and sophomore level calculus, linear algebra, and differential equations and the much more sophisticated senior level mathematics courses taught at Utah. The students are expected to understand definitions and proofs, and the exercises assigned will include proofs as well as computations. The course moves at a leisurely pace, and the material covered includes only Chapters 1, 2, and 3 and selected sections from Chapters 4, 5, and 6. A full year course could easily cover the entire text.

When we teach a one-semester undergraduate course for engineers using these notes, we cover essentially the same material as in the course for mathematics majors, but not all the proofs are done in detail, and there is more emphasis on

computational examples and exercises. The applications in Chapter 6 are given special emphasis.

When taught as a one-semester graduate course, we assume students have some knowledge of complex numbers and so Chapter 1 is given just a brief review. Chapters 2, 3, and 4 are covered completely, parts of Chapters 5 and 6 are covered, along with Chapter 7. If time allows, Chapters 8 and 9 are summarized at the end of the course. In the homework, the emphasis is on the theoretical exercises.

We have tried to present this material in a fashion which is both rigorous and concise, with simple, straightforward explanations. We feel that the modern tendency to expand textbooks with ever more material, excessively verbose explanations, and more and more bells and whistles, simply gets in the way of the student's understanding of the material.

The exercises differ widely in level of abstraction and level of difficulty. They vary from the simple to the quite difficult and from the computational to the theoretical. There are exercises that ask students to prove something or to construct an example with certain properties. There are exercises that ask students to apply theoretical material to help do a computation or to solve a practical problem. Each section contains a number of examples designed to illustrate the material of the section and to teach students how to approach the exercises for that section.

This text, in its various incarnations, has been used by the author and his colleagues for several years at the University of Utah. Each use has led to improvements, additions, and corrections.

The text begins, in Chapter 1, with a discussion of the fact that the real number system is insufficient for some purposes (solving polynomial equations). The system of complex numbers is developed in an attempt to remedy this problem. We then study basic arithmetic of complex numbers, convergence of sequences and series of complex numbers, power series, the exponential function, polar form for complex numbers and the complex logarithm.

The core material of a complex variables course is the material covered here in Chapters 2 and 3. Analytic functions are introduced in Chapter 2 as functions which have a complex derivative. This leads to a discussion of the Cauchy-Riemann equations and harmonic functions. We then introduce contour integrals and the index function (winding number) for closed paths. We prove the Cauchy-integral theorem for triangles and then for convex sets. We believe that this approach leads to the simplest and quickest rigorous proof of a form of Cauchy's theorem that can be used to prove the existence of power series expansions for analytic functions.

The proof that analytic functions have power series expansions occurs early in Chapter 3. This is followed by a wide range of powerful applications with simple proofs – Morera's Theorem, Liouville's Theorem, the Fundamental Theorem of Algebra, the characterization of zeroes and singularities of analytic functions, the Maximum Modulus Principle, and Schwarz's Lemma.

Chapter 4 begins with a proof of the general form of Cauchy's Theorem and Cauchy's Formula. These theorems involve functions which are analytic on a general open set. The integration takes place around a cycle (a generalization of a closed path) which is required to have index zero about any point not in the set. We

go on to study Laurent series, the Residue Theorem, Rouché's Theorem, inverse functions and the Open Mapping Theorem. The chapter ends with a discussion of homotopy and its relationship to integrals around closed paths. We normally do not cover this last section in our undergraduate course, but it would certainly be appropriate for a graduate course using this text.

Residue theory is covered in Chapter 5. We discuss techniques for computing residues as well as a wide variety of applications of residue theory to problems involving the calculation of integrals. We normally do not cover all of this in our undergraduate course. We typically skip the last two sections of this chapter in favor of covering most of Chapter 6.

Chapter 6 deals with conformal maps. We prove the Riemann Mapping Theorem and show how it can be used to transform problems involving analytic or harmonic functions on a simply connected subset of the plane to the analogous problems on the unit disc. We use this technique to study the Dirichlet problem on open, proper, simply connected subsets of $\mathbb{C}$. We discuss applications to heat flow, electrostatics, and hydrodynamics. This material is of particular interest to students in a course on complex variables for engineering students.

The goal of Chapter 7 on analytic continuation is to prove the Picard theorems concerning meromorphic functions at essential singularities. Along the way, we prove the Schwarz Reflection Principle and the Monodromy Theorem, and discuss lifting analytic functions through a covering map.

Chapters 8 and 9 are normally not covered in our one-semester undergraduate complex variables course. Some of the topics in these chapters are, however, often covered in the graduate course. In Chapter 8 we use Weierstrass products to construct an analytic function with a given discrete set of zeroes, with prescribed multiplicities. This is the Weierstrass Theorem. It leads directly to the Weierstrass Factorization Theorem for entire functions. We also prove the Mittag-Leffler Theorem – which gives the existence of a meromorphic function with a prescribed set of poles and principle parts. The final result of the chapter is the proof of Hadamard's Theorem characterizing entire functions of finite order. This is a key ingredient in the proof of the Prime Number Theorem in Chapter 9.

In Chapter 9 we introduce the gamma and zeta functions, develop their basic properties, discuss the Riemann Hypothesis, and prove the Prime Number Theorem. Course notes by our colleague Dragan Miličić provided the original basis for this material. It then went through several years of expansion and refinement before reaching its present form. There is a great deal of technical calculation in this material and we do not cover it in our undergraduate course. However, a few lectures summarizing this material have proved to be a popular way to end the graduate course.

Several standard texts in complex variables and related topics were useful guides in preparing this text. These may also be of interest to the student who wishes to learn more about the subject. They are listed in a short bibliography at the end of the text.