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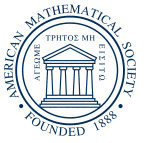
Complex Proofs of Real Theorems

Peter D. Lax
Lawrence Zalcman



American Mathematical Society

http://dx.doi.org/10.1090/ulect/058

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American Mathematical Society
Providence, Rhode Island

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2010 *Mathematics Subject Classification.* Primary 30-XX, 41-XX, 47-XX, 42-XX, 46-XX, 26-XX, 11-XX, 60-XX.

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Library of Congress Cataloging-in-Publication Data

Lax, Peter D.
Complex proofs of real theorems / Peter D. Lax.
p. cm. — (University lecture series ; v. 58)
Includes bibliographical references.
ISBN 978-0-8218-7559-9 (alk. paper)
1. Functions of complex variables. 2. Approximation theory. 3. Functional analysis. I. Zalcman, Lawrence Allen. II. Title.

QA331.7.L39 2012
515.9′53—dc23

2011045859

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To our wives,
Lori and Adrienne

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Preface

At the middle of the twentieth century, the theory of analytic functions of a complex variable occupied an honored, even privileged, position within the canon of core mathematics. This "particularly rich and harmonious theory," averred Hermann Weyl, "is the showpiece of classical nineteenth century analysis."[1] Lest this be mistaken for a gentle hint that the subject was getting old-fashioned, we should recall Weyl's characterization just a few years earlier of Nevanlinna's theory of value distribution for meromorphic functions as "one of the few great mathematical events in our century."[2] Leading researchers in areas far removed from function theory seemingly vied with one another in affirming the "permanent value"[3] of the theory. Thus, Clifford Truesdell declared that "conformal maps and analytic functions will stay current in our culture as long as it lasts";[4] and Eugene Wigner, referring to "the many beautiful theorems in the theory ... of power series and of analytic functions in general," described them as the "most beautiful accomplishments of [the mathematician's] genius."[5] Little wonder, then, that complex function theory was a mainstay of the graduate curriculum, a necessary and integral part of the common culture of all mathematicians.

Much has changed in the past half century, not all of it for the better. From its central position in the curriculum, complex analysis has been pushed to the margins. It is now entirely possible at some institutions to obtain a Ph.D. in mathematics without being exposed to the basic facts of function theory, and (incredible as it may seem) even students specializing in analysis often fulfill degree requirements by taking only a single semester of complex analysis. This, despite the fact that complex variables offers the analyst such indispensable tools as power series, analytic continuation, and the Cauchy integral. Moreover, many important results in real analysis use complex variables in their proofs. Indeed, as Painlevé wrote already at the end of the nineteenth century, "Between two truths of the real domain, the easiest and shortest path quite often passes through the complex

[1]Hermann Weyl, *A half-century of mathematics*, Amer. Math. Monthly **58** (1951), 523-553, p. 526.

[2]Hermann Weyl, *Meromorphic Functions and Analytic Curves*, Princeton University Press, 1943, p. 8.

[3]G. Kreisel, *On the kind of data needed for a theory of proofs*, Logic Colloquium **76**, North Holland, 1977, pp. 111-128, p. 118.

[4]C. Truesdell, *Six Lectures on Modern Natural Philosophy*, Springer-Verlag, 1966, p. 107.

[5]Eugene P. Wigner, *The unreasonable effectiveness of mathematics in the natural sciences*, Comm. Pure Appl. Math. **13** (1960), 1-14, p. 3.

domain,"[6] a claim endorsed and popularized by Hadamard.[7] Our aim in this little book is to illustrate this thesis by bringing together in one volume a variety of mathematical results whose formulations lie outside complex analysis but whose proofs employ the theory of analytic functions. The most famous such example is, of course, the Prime Number Theorem; but, as we show, there are many other examples as well, some of them basic results.

For whom, then, is this book intended? First of all, for everyone who loves analysis and enjoys reading pretty proofs. The technical level is relatively modest. We assume familiarity with basic functional analysis and some elementary facts about the Fourier transform, as presented, for instance, in the first author's *Functional Analysis* (Wiley-Interscience, 2002), referred to henceforth as [**FA**]. In those few instances where we have made use of results not generally covered in the standard first course in complex variables, we have stated them carefully and proved them in appendices. Thus the material should be accessible to graduate students. A second audience consists of instructors of complex variable courses interested in enriching their lectures with examples which use the theory to solve problems drawn from outside the field.

Here is a brief summary of the material covered in this volume. We begin with a short account of how complex variables yields quick and efficient solutions of two problems which were of great interest in the seventeenth and eighteenth centuries, viz., the evaluation of $\sum_1^\infty 1/n^2$ and related sums and the proof that every algebraic equation in a single variable (with real or even complex coefficients) is solvable in the field of complex numbers. Next, we discuss two representative applications of complex analysis to approximation theory in the real domain: weighted polynomial approximation on the line and uniform approximation on the unit interval by linear combinations of the functions $\{x^{n_k}\}$, where $n_k \to \infty$ (Müntz's Theorem). We then turn to applications of complex variables to operator theory and harmonic analysis. These chapters form the heart of the book. A first application to operator theory is Rosenblum's elegant proof of the Fuglede-Putnam Theorem. We then discuss Toeplitz operators and their inversion, Beurling's characterization of the invariant subspaces of the unilateral shift on the Hardy space H^2 and the consequent divisibility theory for the algebra $\mathcal{B}$ of bounded analytic functions on the disk or half-plane, and a celebrated problem in prediction theory (Szegő's Theorem). We also prove the Riesz-Thorin Convexity Theorem and use it to deduce the boundedness of the Hilbert transform on $L^p(\mathbb{R})$, $1 < p < \infty$. The chapter on applications to harmonic analysis begins with D.J. Newman's striking proof of Fourier uniqueness via complex variables; continues on to a discussion of a curious functional equation and questions of uniqueness (and nonuniqueness) for the Radon transform; and then turns to the Paley-Wiener Theorem, which together with the divisibility theory for $\mathcal{B}$ referred to above is exploited to provide a simple proof of the Titchmarsh Convolution Theorem. This chapter concludes with Hardy's Theorem, which quantifies the fact that a function and its Fourier transform cannot both tend to zero

[6]"Entre deux vérités du domain réel, le chemin le plus facile et le plus court passe bien souvent par le domaine complexe." Paul Painlevé, *Analyse des travaux scientifiques*, Gauthier-Villars, 1900, pp.1-2.

[7]"It has been written that the shortest and best way between two truths of the real domain often passes through the imaginary one." Jacques Hadamard, *An Essay on the Psychology of Invention in the Mathematical Field*, Princeton University Press, 1945, p. 123.

too rapidly. The final chapters are devoted to the Gleason-Kahane-Żelazko Theorem (in a unital Banach algebra, a subspace of codimension 1 which contains no invertible elements is a maximal ideal) and the Fatou-Julia-Baker Theorem (the Julia set of a rational function of degree at least 2 or a nonlinear entire function is the closure of the repelling periodic points). We end on a high note, with a proof of the Prime Number Theorem. A coda deals very briefly with two unusual applications: one to fluid dynamics (the design of shockless airfoils for partly supersonic flows), and the other to statistical mechanics (the stochastic Loewner evolution).

To a certain extent, the choice of topics is canonical; but, inevitably, it has also been influenced by our own research interests. Some of the material has been adapted from [**FA**]. Our title echoes that of a paper by the second author.[8]

Although this book has been in the planning stages for some time, the actual writing was done during the Spring and Summer of 2010, while the second author was on sabbatical from Bar-Ilan University. He thanks the Courant Institute of Mathematical Sciences of New York University for its hospitality during part of this period and acknowledges the support of Israel Science Foundation Grant 395/07.

Finally, it is a pleasure to acknowledge valuable input from a number of friends and colleagues. Charles Horowitz read the initial draft and made many useful comments. David Armitage, Walter Bergweiler, Alex Eremenko, Aimo Hinkkanen, and Tony O'Farrell all offered perceptive remarks and helpful advice on subsequent versions. Special thanks to Miriam Beller for her expert preparation of the manuscript.

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[8]Lawrence Zalcman, *Real proofs of complex theorems (and vice versa)*, Amer. Math. Monthly **81** (1974), 115-137.

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Complex Proofs of Real Theorems is an extended meditation on Hadamard's famous dictum, "The shortest and best way between two truths of the real domain often passes through the imaginary one." Directed at an audience acquainted with analysis at the first year graduate level, it aims at illustrating how complex variables can be used to provide quick and efficient proofs of a wide variety of important results in such areas of analysis as approximation theory, operator theory, harmonic analysis, and complex dynamics.

Topics discussed include weighted approximation on the line, Müntz's theorem, Toeplitz operators, Beurling's theorem on the invariant spaces of the shift operator, prediction theory, the Riesz convexity theorem, the Paley–Wiener theorem, the Titchmarsh convolution theorem, the Gleason–Kahane–Żelazko theorem, and the Fatou–Julia–Baker theorem. The discussion begins with the world's shortest proof of the fundamental theorem of algebra and concludes with Newman's almost effortless proof of the prime number theorem. Four brief appendices provide all necessary background in complex analysis beyond the standard first year graduate course. Lovers of analysis and beautiful proofs will read and reread this slim volume with pleasure and profit.

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