# University <br> LECTURE Series 

Volume 57

# Real Solutions to Equations from Geometry 

Frank Sottile



## Real Solutions to Equations from Geometry

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# Real Solutions <br> to Equations <br> from Geometry 

Frank Sottile



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Dedicated to the memory of my first teacher, Samuel Sottile, who died as I began these notes in 2005

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## Preface

Understanding, finding, or even deciding on the existence of real solutions to a system of equations is a very difficult problem with many applications outside of mathematics. While it is hopeless to expect much in general, we know a surprising amount about these questions for systems which possess additional structure coming from geometry. Such equations from geometry for which we have information about their real solutions are the subject of this book.

This book focuses on equations from toric varieties and Grassmannians. Not only is much known in these cases, but they encompass some of the most common applications. The results may be grouped into three themes:
(I) Upper bounds on the number of real solutions.
(II) Geometric problems that can have all solutions be real.
(III) Lower bounds on the number of real solutions.

Upper bounds (I) bound the complexity of the set of real solutions - they are one of the sources for the theory of o-minimal structures which are an important topic in real algebraic geometry. The existence (II) of geometric problems that can have all solutions be real was initially surprising, but this phenomenon now appears to be ubiquitous. Lower bounds (III) give existence proofs of real solutions. Their most spectacular manifestation is the nontriviality of the Welschinger invariant, which was computed via tropical geometry. One of the most surprising manifestations of this phenomenon is when the upper bound equals the lower bound, which is the subject of the Shapiro Conjecture and the focus of the last five chapters.

I thank the Institut Henri Poincaré, where a preliminary version of these notes was produced during a course I taught in November 2005. These notes were revised and expanded during courses at Texas A\&M University in 2007 and in 2010 and at a lecture series at the Centre Interfacultaire Bernoulli at EPFL in 2008 and were completed in 2011 while in residence at the Institut Mittag-Leffler with material from a lecture at the January 2009 Joint Mathematics Meetings on the Theorem of Mukhin, Tarasov, and Varchenko and from lectures at the GAeL conference in Leiden in June 2009. I also thank Prof. Dr. Peter Gritzmann of the Technische Universität München, whose hospitality enabled the completion of the first version of these notes. During this period, this research was supported by NSF grants DMS-1001615, DMS-0701059, and CAREER grant DMS-0538734. The point of view in these notes was developed through the encouragement and inspiration of Bernd Sturmfels, Askold Khovanskii, Maurice Rojas, and Marie-Françoise Roy and through my interactions with the many people whose work is mentioned here, including my collaborators from whom I have learned a great deal.

Frank Sottile
04.25.11, Djursholm, Sweden

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## Index of Notation

$\#_{m, p}$, Schubert number, 5
$-\rightarrow$, rational map, 79
$\lessdot$, cover in a poset, 91,126
$\mathcal{A}$, support of a polynomial, 3,26
$\mathcal{A}^{+}$, lift of $\mathcal{A}, 30$
$\alpha(\kappa)$, sequence associated to a subset $\kappa \subset[m], 174$
$\beta(\kappa)$, sequence associated to a subset $\kappa \subset[m], 175$
$\binom{[m+p]}{p}$, Bruhat order, 97
$\square$, Schubert condition $(m, m+2, \ldots, m+p)$, 117, 122
$\mathbb{C}$, complex numbers, 1
$\mathbb{C}^{*}$, nonzero complex numbers, 1
$C_{m, p}$, product of chains, 96
$\mathbb{C}\{t\}$, field of Puiseaux series, 34
$\mathbb{C}[X]$, homogeneous coordinate ring of projective variety $X, 30$
$d(\mathcal{A})$, number of solutions to system with support $\mathcal{A}, 26$
$d(\alpha)$, degree of Schubert variety $X_{\alpha}, 128$
$\operatorname{deg}(X)$, degree of subvariety $X, 29$
$\Delta_{\mathcal{A}}$, convex hull of $\mathcal{A}, 3,26$
$\Delta_{\omega}$, regular polyhedral subdivision induced by $\omega, 42$
$\Delta_{p}$, positive chamber of hyperplane complement, 65
$\operatorname{des}(w)$, descent of Grassmannian condition $w, 183$
$d\left(f_{1}, \ldots, f_{n}\right)$, number of solutions to $f_{1}=\cdots=f_{n}=0,26$
$\mathbf{e}_{\alpha}$, basis element of $\wedge^{p} \mathbb{C}^{m+p}, 96$
$E_{\bullet}(s)$, flag of polynomials vanishing to different orders at $s, 131$
$F_{\bullet}$, flag of subspaces, 117
$F_{\bullet}(t)$, osculating flag, 119, 121
$\mathbb{F} \ell(\mathbf{a} ; m)$, flag manifold, 180
$G$, linear algebraic group, 179
$g^{+}$, lift of map $g$ to sphere, 80
$G / P$, flag manifold, 179
$\gamma$, rational normal curve, $6,118,121$
$\operatorname{Gr}\left(m, \mathbb{C}_{m+p-1}[t]\right)$, Grassmannian, 5
$\mathrm{Gr}(p, m+p)$, Grassmannian, 96,116
$\mathcal{H}$, hyperplane arrangement, 63
$H_{X}(d)$, Hilbert function of projective variety $X, 31$
$h_{X}(d)$, Hilbert polynomial, 30
$I_{m, p}$, Plücker ideal, 98
$\mathrm{in}_{\omega}(F)$, facial system, 33
$\operatorname{in}_{\omega}(f)$, initial form, 33, 40
$\operatorname{in}_{\omega}\left(X_{\mathcal{A}}\right)$, initial scheme, 40
$K_{\mathbf{a}}$, Kostka number, 143
$\|\kappa\|=\kappa_{1}+\cdots+\kappa_{k}$, sum of a sequence, 174
$\lambda(P)$, number of linear extensions of poset $P, 92$
LG( $m$ ), Lagrangian Grassmannian, 175
$\lg (w)$, length of a permutation, 103, 179
$\operatorname{mdeg}(\rho)$, mapping degree of $\rho, 19$
$M_{\mathcal{H}}$, complement of hyperplane arrangement, 63
$m(\omega, \mathcal{A})$, minimum value of $\omega$ on $\mathcal{A}, 33$
$\mathrm{MV}\left(K_{1}, \ldots, K_{n}\right)$, mixed volume of convex bodies $K_{1}, \ldots, K_{n}, 3,27$
$\mathbb{N}$, natural numbers, 1
$[n]$, set of integers $\{1,2, \ldots, n\}, 2$
$\mathrm{OG}(m)$, orthogonal Grassmannian, 173
$\mathcal{O}_{P}$, order polytope of poset $P, 92$
$\mathbb{P}^{\mathcal{A}}$, projective space with coordinates indexed by $\mathcal{A}, 27$
$\varphi_{\mathcal{A}}$, monomial parameterization, 27
$\mathbb{P}^{n}$, complex projective space, 2
$P_{\omega}$, lifted polytope, 42
$\psi$, affine-linear map, 63
$\mathbb{Q}$, rational numbers, 1
$\mathbb{R}$, real numbers, 1
$\mathbb{R}^{*}$, nonzero real numbers, 1
$\mathbb{R}_{>}$, positive real numbers, 1,65
$\mathbb{R}_{>}^{n}$, positive orthant, 49
$\rho$, rational function, 19
$\mathcal{R}_{p+1}$, real rational functions of degree $p+1$ with only real critical points, 135
$\mathbb{R} \mathbb{P}_{>0}^{\mathcal{A}}$, nonnegative orthant, 83
$\mathbb{R P}^{\bar{n}}$, real projective space, 2
$\sigma_{m, p}$, degree of real Wronski map, 10
$\sigma(\omega)$, signature of a foldable triangulation, 85
$\sigma(P)$, sign-imbalance of poset $P, 92$
$\operatorname{sign}(w)$, sign of permutation $w, 93$
$S_{N}$, symmetric group, 149
$\mathbb{S}^{n}, n$-dimensional sphere, 80
$\mathcal{S}_{\omega}$, regular subdivision, 42
$\mathrm{St}(p, m+p)$, Stiefel manifold, 96
$\mathrm{St}_{\mathbb{R}}(2, p+1)$, real Stiefel manifold, 134
$\mathbb{T}$, nonzero complex numbers $\mathbb{C}^{*}, 1,3,26$
$\mathbb{T}_{\mathbb{R}}$, nonzero real numbers $\mathbb{R}^{*}, 1,65$
$T_{x} X$, tangent space of $X$ at $x, 11$
$\mathrm{ubc}_{D}(C)$, number of unbounded components of curve $C, 67$
$V_{D}\left(g_{1}, \ldots, g_{m}\right)$, common zeroes of $g_{i}$ in $D$, 67
$\operatorname{var}(c)$, variation in a sequence $c, 14$
$\operatorname{var}(F, a)$, variation in a sequence $F$ of polynomials at $a \in \mathbb{R}, 14$
$V_{\mu}$, highest weight module, 153
volume $(\Delta)$, volume of polytope $\Delta, 3,26$
$W_{\kappa, c}(x)$, Wronski polynomial, 85
Wr, Wronski map, 5
$\operatorname{Wr}\left(f_{1}, \ldots, f_{m}\right)$, Wronskian of $f_{1}, \ldots, f_{m}, 5$
$\mathrm{Wr}_{\mathbb{R}}$, real Wronski map, 11
$X_{\mathcal{A}}$, toric variety, 29
$\mathcal{X}_{\mathcal{A}}$, toric degeneration, 40
$x^{a}$, monomial, 3
$X_{\alpha} F_{\bullet}$, Schubert variety, 117
$X_{\alpha}^{\circ} F_{\bullet}$, Schubert cell, 126
$X(l, n)$, Khovanskii number, 4, 49, 50
$X_{\mathbb{R}}(l, n)$, Khovanskii number, 50
$X_{w} B$, Schubert variety, 179
$Y_{\mathcal{A}}$, real part of toric variety $X_{\mathcal{A}}, 79$
$Y_{\mathcal{A}}^{+}$, spherical toric variety, 80
$Y_{\mathcal{A},>}$, positive part of toric variety $X_{\mathcal{A}}, 88$
$\mathbb{Z}$, integers, 1
$\mathbb{Z} \mathcal{A}$, integer affine span of $\mathcal{A}, 28$
$\mathbb{Z}^{n}$, integer lattice, 2
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