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Dedicated to the memory of my first teacher, Samuel Sottile, who died as I began these notes in 2005

Contents

Preface

Understanding, finding, or even deciding on the existence of real solutions to a system of equations is a very difficult problem with many applications outside of mathematics. While it is hopeless to expect much in general, we know a surprising amount about these questions for systems which possess additional structure coming from geometry. Such equations from geometry for which we have information about their real solutions are the subject of this book.

This book focuses on equations from toric varieties and Grassmannians. Not only is much known in these cases, but they encompass some of the most common applications. The results may be grouped into three themes:

(I) Upper bounds on the number of real solutions.
(II) Geometric problems that can have all solutions be real.
(III) Lower bounds on the number of real solutions.

Upper bounds (I) bound the complexity of the set of real solutions—they are one of the sources for the theory of o-minimal structures which are an important topic in real algebraic geometry. The existence (II) of geometric problems that can have all solutions be real was initially surprising, but this phenomenon now appears to be ubiquitous. Lower bounds (III) give existence proofs of real solutions. Their most spectacular manifestation is the nontriviality of the Welschinger invariant, which was computed via tropical geometry. One of the most surprising manifestations of this phenomenon is when the upper bound equals the lower bound, which is the subject of the Shapiro Conjecture and the focus of the last five chapters.

I thank the Institut Henri Poincaré, where a preliminary version of these notes was produced during a course I taught in November 2005. These notes were revised and expanded during courses at Texas A&M University in 2007 and in 2010 and at a lecture series at the Centre Interfacultaire Bernoulli at EPFL in 2008 and were completed in 2011 while in residence at the Institut Mittag-Leffler with material from a lecture at the January 2009 Joint Mathematics Meetings on the Theorem of Mukhin, Tarasov, and Varchenko and from lectures at the GAeL conference in Leiden in June 2009. I also thank Prof. Dr. Peter Gritzmann of the Technische Universität München, whose hospitality enabled the completion of the first version of these notes. During this period, this research was supported by NSF grants DMS-1001615, DMS-0701059, and CAREER grant DMS-0538734. The point of view in these notes was developed through the encouragement and inspiration of Bernd Sturmfels, Askold Khovanskii, Maurice Rojas, and Marie-Françoise Roy and through my interactions with the many people whose work is mentioned here, including my collaborators from whom I have learned a great deal.

Frank Sottile
04.25.11, Djursholm, Sweden

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Understanding, finding, or even deciding on the existence of real solutions to a system of equations is a difficult problem with many applications outside of mathematics. While it is hopeless to expect much in general, we know a surprising amount about these questions for systems which possess additional structure often coming from geometry.

Photo; Maria Sottile, 2011.

This book focuses on equations from toric varieties and Grassmannians. Not only is much known about these, but such equations are common in applications. There are three main themes: upper bounds on the number of real solutions, lower bounds on the number of real solutions, and geometric problems that can have all solutions be real. The book begins with an overview, giving background on real solutions to univariate polynomials and the geometry of sparse polynomial systems. The first half of the book concludes with fewnomial upper bounds and with lower bounds to sparse polynomial systems. The second half of the book begins by sampling some geometric problems for which all solutions can be real, before devoting the last five chapters to the Shapiro Conjecture, in which the relevant polynomial systems have only real solutions.

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