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Michael Gekhtman
Michael Shapiro
Alek Vainshtein

American Mathematical Society

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Providence, Rhode Island

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Dedicated to the memory of Vladimir Igorevich Arnold

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Preface

Cluster algebras introduced by Fomin and Zelevinsky in [**FZ2**] are commutative rings with unit and no zero divisors equipped with a distinguished family of generators (*cluster variables*) grouped in overlapping subsets (*clusters*) of the same cardinality (*the rank of the cluster algebra*) connected by *exchange relations*. Among these algebras one finds coordinate rings of many algebraic varieties that play a prominent role in representation theory, invariant theory, the study of total positivity, etc. For instance, homogeneous coordinate rings of Grassmannians, coordinate rings of simply-connected connected semisimple groups, Schubert varieties, and other related varieties carry (possibly after a small adjustment) a cluster algebra structure. A prototypic example of exchange relations is given by the famous Plücker relations, and precursors of the cluster algebra theory one can observe in the Ptolemy theorem expressing product of diagonals of inscribed quadrilateral in terms of side lengths and in the Gauss formulae describing Pentagrama Myrificum.

Cluster algebras were introduced in an attempt to create an algebraic and combinatorial framework for the study dual canonical bases and total positivity in semisimple groups. The notion of *canonical bases* introduced by Lusztig for quantized enveloping algebras plays an important role in the representation theory of such algebras. One of the approaches to the description of canonical bases utilizes dual objects called *dual canonical bases*. Namely, elements of the enveloping algebra are considered as differential operators on the space of functions on the corresponding group. Therefore, the space of functions is considered as the dual object, whereas the pairing between a differential operator D and a function F is defined in the standard way as the value of DF at the unity. Elements of dual canonical bases possess special positivity properties. For instance, they are regular positive-valued functions on the so-called *totally positive* subvarieties in reductive Lie groups, first studied by Lusztig. In the case of GL_n the notion of total positivity coincides with the classical one, first introduced by Gantmakher and Krein: a matrix is totally positive if all of its minors are positive. Certain finite collections of elements of dual canonical bases form distinguished coordinate charts on totally positive varieties. The positivity property of the elements of dual canonical bases and explicit expressions for these collections and transformations between them were among the sources of inspiration for designing cluster algebra transformation mechanism. Transitions between distinguished charts can be accomplished via sequences of relatively simple positivity preserving transformations that served as a model for an abstract definition of a cluster transformation. Cluster algebra transformations construct new distinguished elements of the cluster algebra from the initial collection of elements. In many concrete situations all constructed distinguished elements have certain stronger positivity properties called *Laurent*

positivity. It is still an open question whether Laurent positivity holds for all distinguished elements for an arbitrary cluster algebra. All Laurent positive elements of a cluster algebra form a cone. Conjecturally, for cluster algebras arising from reductive semisimple Lie groups extremal rays of this cone form a basis closely connected to the dual canonical basis mentioned above.

Since then, the theory of cluster algebras has witnessed a spectacular growth, first and foremost due to the many links that have been discovered with a wide range of subjects including

- representation theory of quivers and finite-dimensional algebras and categorification;
- discrete dynamical systems based on rational recurrences, in particular, Y-systems in the thermodynamic Bethe Ansatz;
- Teichmüller and higher Teichmüller spaces;
- combinatorics and the study of combinatorial polyhedra, such as the Stasheff associahedron and its generalizations;
- commutative and non-commutative algebraic geometry, in particular,
  - Grassmannians, projective configurations and their tropical analogues,
  - the study of stability conditions in the sense of Bridgeland,
  - Calabi-Yau algebras,
  - Donaldson-Thomas invariants,
  - moduli space of (stable) quiver representations.

In this book, however, we deal only with one aspect of the cluster algebra theory: its relations to Poisson geometry and theory of integrable systems. First of all, we show that the cluster algebra structure, which is purely algebraic in its nature, is closely related to certain Poisson (or, dually, pre-symplectic) structures. In the cases of double Bruhat cells and Grassmannians discussed below, the corresponding families of Poisson structures include, among others, standard R-matrix Poisson-Lie structures (or their push-forwards). A large part of the book is devoted to the interplay between cluster structures and Poisson/pre-symplectic structures. This leads, in particular, to revealing of cluster structure related to integrable systems called Toda lattices and to dynamical interpretation of cluster transformations, see the last chapter. Vice versa, Poisson/pre-symplectic structures turned out to be instrumental for the proof of purely algebraic results in the general theory of cluster algebras.

In Chapter 1 we introduce necessary notions and notation. Section 1.1 provides a very concise introduction to *flag varieties*, *Grassmannians* and *Plücker coordinates*. Section 1.2 treats *simple Lie algebras* and *groups*. Here we remind to the reader the standard objects and constructions used in Lie theory, including the *adjoint action*, the *Killing form*, *Cartan subalgebras*, *root systems*, and *Dynkin diagrams*. We discuss in some detail *Bruhat decompositions* of a simple Lie group, and *double Bruhat cells*, which feature prominently in the next Chapter. *Poisson–Lie groups* are introduced in Section 1.3. We start with providing basic definitions of Poisson geometry, and proceed to define main objects in Poisson–Lie theory, including the *classical R-matrix*. *Sklyanin brackets* are then defined as a particular example of an R-matrix Poisson bracket. Finally, we treat in some detail the case of the *standard Poisson–Lie structure* on a simple Lie group, which will play an important role in subsequent chapters.

Chapter 2 considers in detail two basic sources of cluster-like structures in rings of functions related to Schubert varieties: the homogeneous coordinate ring of the Grassmannian $G_2(m)$ of 2-dimensional planes and the ring of regular functions on a double Bruhat cell. The first of the two rings is studied in Section 2.1. We show that Plücker coordinates of $G_2(m)$ can be organized into a finite number of groups of the same cardinality (*clusters*), covering the set of all Plücker coordinates. Every cluster corresponds to a triangulation of a convex m-gon. The system of clusters has a natural graph structure, so that adjacent clusters differ exactly by two Plücker coordinates corresponding to a pair of crossing diagonals (and contained together in exactly one short Plücker relation). Moreover, this graph is a 1-skeleton of the Stasheff polytope of m-gon triangulations. We proceed to show that if one fixes an arbitrary cluster, any other Plücker coordinate can be expressed as a rational function in the Plücker coordinates entering this cluster. We prove that this rational function is a Laurent polynomial and find a geometric meaning for its numerator and denominator, see Proposition 2.1.

Section 2.2 starts with the formulation of Arnold's problem: find the number of connected components in the variety of real complete flags intersecting transversally a given pair of flags. We reformulate this problem as a problem of enumerating connected components in the intersection of two real open Schubert cells, and proceed to a more general problem of enumerating connected components of a real double Bruhat cell. It is proved that components in question are in a bijection with the orbits of a group generated by symplectic transvections in a vector space over the field $\mathbb{F}_2$, see Theorem 2.10. As one of the main ingredients of the proof, we provide a complete description of the ring of regular functions on the double Bruhat cell. It turns out that generators of this ring can be grouped into clusters, and that they satisfy Plücker-type exchange relations.

In Chapter 3 we introduce cluster algebras and prove two fundamental results about them. Section 3.1 contains basic definitions and examples. In this book we mainly concentrate on *cluster algebras of geometric type*, so the discussion in this Section is restricted to such algebras, and the case of general coefficients is only mentioned in Remark 3.13. We define basic notions of *cluster* and *stable variables*, *seeds*, *exchange relations*, *exchange matrices* and their *mutations*, *exchange graphs*, and provide extensive examples. The famous *Laurent phenomenon* is treated in Section 3.2. We prove both the general statement (Theorem 3.14) and its sharpening for cluster algebras of geometric type (Proposition 3.20). The second fundamental result, the classification of cluster algebras of *finite type*, is discussed in Section 3.3. We state the result (Theorem 3.26) as an equivalence of three conditions, and provide complete proofs for two of the three implications. The third implication is discussed only briefly, since a complete proof would require exploring intricate combinatorial properties of root systems for different Cartan–Killing types, which goes beyond the scope of this book. In Section 3.4 we discuss relations between cluster algebras and *rings of regular functions*, see Proposition 3.37. Finally, Section 3.5 contains a list of conjectures, some of which are treated in subsequent chapters.

Chapter 4 is central to the book. In Section 4.1 we introduce the notion of Poisson brackets *compatible* with a cluster algebra structure and provide a complete characterization of such brackets for cluster algebras with the extended exchange matrix of full rank, see Theorem 4.5. In this context, mutations of the exchange matrix are explained as transformations of the coefficient matrix of the compatible

Poisson bracket induced by a basis change. In Section 4.2 we apply this result to the study of Poisson and cluster algebra structures on Grassmannians. Starting from the Sklyanin bracket on SL_n, we define the corresponding Poisson bracket on the open Schubert cell in the Grassmannian $G_k(n)$ and construct the cluster algebra compatible with this Poisson bracket. It turns out that this cluster algebra is isomorphic to the ring of regular functions on the open cell in $G_k(n)$, see Theorem 4.14. We further investigate this construction and prove that an extension of the obtained cluster algebra is isomorphic to the homogeneous coordinate ring of the Grassmannian, see Theorem 4.17.

The smooth part of the spectrum of a cluster algebra is called the *cluster manifold* and is treated in Chapter 5. The definition of the cluster manifold is discussed in Section 5.1. In Section 5.2 we investigate the natural *Poisson toric action* on the cluster manifold and provide necessary and sufficient conditions for the extendability of the local toric action to the global one, see Lemma 5.3. We proceed to the enumeration of connected components of the regular locus of the toric action in Section 5.3 and extend Theorem 2.10 to this situation, see Theorem 5.9. In Section 5.4 we study the structure of the regular locus and prove that it is foliated into disjoint union of generic symplectic leaves of the compatible Poisson bracket, see Theorem 5.12. In Section 5.5 we apply these results to the enumeration of connected components in the intersection of n Schubert cells in general position in $G_k(n)$, see Theorem 5.15.

Note that compatible Poisson brackets are defined in Chapter 4 only for cluster algebras with the extended exchange matrix of a full rank. To overcome this restriction, we present in Chapter 6 a dual approach based on *pre-symplectic* rather than on Poisson structures. In Section 6.1 we define closed 2-forms *compatible* with a cluster algebra structure and provide a complete characterization of such forms parallel to Theorem 4.5, see Theorem 6.2. Further, we define the *secondary cluster manifold* and a compatible symplectic form on it, which we call the *Weil–Petersson form* associated to the cluster algebra. The reason for such a name is explained in Section 6.2, which treats our main example, the *Teichmüller space*. We briefly discuss *Penner coordinates* on the decorated Teichmüller space defined by fixing a triangulation of a Riemann surface Σ, and observing that Ptolemy relations for these coordinates can be considered as exchange relations. The secondary cluster manifold for the cluster algebra $\mathcal{A}(\Sigma)$ arising in this way is the Teichmüller space, and the Weil–Petersson form associated with this cluster algebra coincides with the classical Weil–Petersson form corresponding to Σ, see Theorem 6.6. We proceed with providing a geometric meaning for the degrees of variables in the denominators of Laurent polynomials expressing arbitrary cluster variables in terms of the initial cluster, see Theorem 6.7; Proposition 2.1 can be considered as a toy version of this result. Finally, we give a geometric description of $\mathcal{A}(\Sigma)$ in terms of triangulation equipped with a *spin*, see Theorem 6.9. In Section 6.3 we derive an axiomatic approach to exchange relations. We show that the class of transformations satisfying a number of natural conditions, including the compatibility with a closed 2-form, is very restricted, and that exchange transformations used in cluster algebras are simplest representatives of this class, see Theorem 6.11.

In Chapter 7 we apply the results of previous chapters to prove several conjectures about *exchange graphs* of cluster algebras listed in Section 3.5. The dependence of a general cluster algebra on the coefficients is investigated in Section 7.1.

We prove that the exchange graph of the cluster algebra with principal coefficients covers the exchange graph of any other cluster algebra with the same exchange matrix, see Theorem 7.1. In Section 7.2 we consider vertices and edges of an exchange graph and prove that distinct seeds have distinct clusters for cluster algebras of geometric type and for cluster algebras with arbitrary coefficients and a non-degenerate exchange matrix, see Theorem 7.4. Besides, we prove that if a cluster algebra has the above cluster-defines-seed property, then adjacent vertices of its exchange graph are clusters that differ only in one variable, see Theorem 7.5. Finally, in Section 7.3 we prove that the exchange graph of a cluster algebra with a non-degenerate exchange matrix does not depend on coefficients, see Theorem 7.7.

In the remaining three chapters we develop an approach to the interaction of Poisson and cluster structures based on the study of *perfect networks*—directed networks on surfaces with boundary having trivalent interior vertices and univalent boundary vertices. Perfect planar networks in the *disk* are treated in Chapter 8. Main definitions, including *weights* and *boundary measurements*, are given in Section 8.1. We prove that each boundary measurement is a rational function of the edge weights, see Proposition 8.3, and define the *boundary measurement map* from the space of edge weights of the network to the space of $k \times m$ matrices, where k and m are the numbers of sources and sinks. Section 8.2 is central to this chapter; it treats Poisson structures on the space of edge weights of the network and Poisson structures on the space of $k \times m$ matrices induced by the boundary measurement map. The former are defined axiomatically as satisfying certain natural conditions, including an analog of the Poisson–Lie property for groups. It is proved that such structures form a 6-parametric family, see Proposition 8.5. For fixed sets of sources and sinks, the induced Poisson structures on the space of $k \times m$ matrices form a 2-parametric family that does not depend on the internal structure of the network, see Theorem 8.6. Explicit expressions for this 2-parametric family are provided in Theorem 8.7. In case $k = m$ and separated sources and sinks, we recover on $k \times k$ matrices the Sklyanin bracket corresponding to a 2-parametric family of classical R-matrices including the standard R-matrix, see Theorem 8.10. In Section 8.3 we extend the boundary measurement map to the Grassmannian $G_k(k + m)$ and prove that the obtained Grassmannian boundary measurement map induces a 2-parametric family of Poisson structures on $G_k(k+m)$ that does not depend on the particular choice of k sources and m sinks; moreover, for any such choice, the family of Poisson structures on $k \times m$ matrices representing the corresponding cell of $G_k(k+m)$ coincides with the family described in Theorem 8.7 (see Theorem 8.12 for details). Next, we give an interpretation of the natural GL_{k+m} action on $G_k(k+m)$ in terms of networks and establish that every member of the above 2-parametric family of Poisson structures on $G_k(k + m)$ makes it into a Poisson homogeneous space of GL_{k+m} equipped with the Sklyanin R-matrix bracket, see Theorem 8.17. Finally, in Section 8.4 we prove that each bracket in this family is compatible with the cluster algebra constructed in Chapter 4, see Theorem 8.20. An important ingredient of the proof is the use of face weights instead of edge weights.

In Chapter 9 we extend the constructions of the previous chapter to perfect networks in an *annulus*. Section 9.1.1 is parallel to Section 8.1; the main difference is that in order to define boundary measurements we have to introduce an auxiliary parameter λ that counts intersections of paths in the network with a *cut* whose endpoints belong to distinct boundary circles. We prove that boundary measurements

are rational functions in the edge weights and λ, see Corollary 9.3 and study how they depend on the choice of the cut. Besides, we provide a *cohomological description* of the space of face and trail weights, which is a higher analog of the space of edge weights used in the previous chapter. Poisson properties of the obtained boundary measurement map from the space of edge weights to the space of rational $k \times m$ matrix functions in one variable are treated in Section 9.2. We prove an analog of Theorem 8.6, saying that for fixed sets of sources and sinks, the induced Poisson structures on the space of matrix functions form a 2-parametric family that does not depend on the internal structure of the network, see Theorem 9.4. Explicit expressions for this family are much more complicated than those for the disk, see Proposition 9.6. The proof itself differs substantially from the proof of Theorem 8.6. It relies on the fact that any rational $k \times m$ matrix function belongs to the image of the boundary measurement map for an appropriated perfect network, see Theorem 9.10. The section is concluded with Theorem 9.15 claiming that for a specific choice of sinks and sources one can recover the Sklyanin bracket corresponding to the trigonometric R-matrix. In Section 9.3 we extend these results to the *Grassmannian boundary measurement map* from the space of edge weights to the space of Grassmannian loops. We define the *path reversal map* and prove that this map commutes with the Grassmannian boundary measurement map, see Theorem 9.17. Further, we prove Theorem 9.22, which is a natural extension of Theorem 8.12; once again, the proof is very different and is based on path reversal techniques and the use of face weights.

In the concluding Chapter 10 we apply techniques developed in the previous chapter to providing a cluster interpretation of *generalized Bäcklund–Darboux transformations* for *Coxeter–Toda lattices*. Section 10.1 gives an overview of the chapter and contains brief introductory information on Toda lattices, *Weyl functions* of the corresponding *Lax operators* and generalized Bäcklund–Darboux transformations between phase spaces of different lattices preserving the Weyl function. A *Coxeter double Bruhat cell* in GL_n is defined by a pair of Coxeter elements in the symmetric group S_n. In Section 10.2 we have collected and proved all the necessary technical facts about such cells and the representation of their elements via perfect networks in a disk. Section 10.3 treats the inverse problem of restoring factorization parameters of an element of a Coxeter double Bruhat cell from its Weyl function. We provide an explicit solution for this problem involving *Hankel determinants* in the coefficients of the Laurent expansion for the Weyl function, see Theorem 10.9. In Section 10.4 we build and investigate a cluster algebra on a certain space $\mathcal{R}_n$ of rational functions related to the space of Weyl functions corresponding to Coxeter double Bruhat cells. We start from defining a perfect network in an annulus corresponding to the network in a disk studied in Section 10.2. The space of face weights of this network is equipped with a particular Poisson bracket from the family studied in Chapter 9. We proceed by using results of Chapter 4 to build a cluster algebra of rank $2n-2$ compatible with this Poisson bracket. Theorem 10.27 claims that this cluster algebra does not depend on the choice of the pair of Coxeter elements, and that the ring of regular functions on $\mathcal{R}_n$ is isomorphic to the localization of this cluster algebra with respect to the stable variables. In Section 10.5 we use these results to characterize generalized Bäcklund–Darboux transformations as sequences of cluster transformations in the above cluster algebra conjugated by by a certain map defined by the solution of the inverse problem in Section 10.3, see Theorem 10.36.

We also show how one can interpret generalized Bäcklund–Darboux transformations via equivalent transformations of the corresponding perfect networks in an annulus. In conclusion, we explain that classical Darboux transformations can be also interpreted via cluster transformations, see Proposition 10.39.

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Cluster algebras, introduced by Fomin and Zelevinsky in 2001, are commutative rings with unit and no zero divisors equipped with a distinguished family of generators (cluster variables) grouped in overlapping subsets (clusters) of the same cardinality (the rank of the cluster algebra) connected by exchange relations. Examples of cluster algebras include coordinate rings of many algebraic varieties that play a prominent role in representation theory, invariant theory, the study of total positivity, etc. The theory of cluster algebras has witnessed a spectacular growth, first and foremost due to the many links to a wide range of subjects including representation theory, discrete dynamical systems, Teichmüller theory, and commutative and non-commutative algebraic geometry.

This book is the first devoted to cluster algebras. After presenting the necessary introductory material about Poisson geometry and Schubert varieties in the first two chapters, the authors introduce cluster algebras and prove their main properties in Chapter 3. This chapter can be viewed as a primer on the theory of cluster algebras. In the remaining chapters, the emphasis is made on geometric aspects of the cluster algebra theory, in particular on its relations to Poisson geometry and to the theory of integrable systems.

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