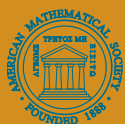


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Volume 159

Potential Theory and Dynamics on the Berkovich Projective Line

Matthew Baker
Robert Rumely



American Mathematical Society

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Providence, Rhode Island

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Preface

This book is a revised and expanded version of the authors' manuscript "Analysis and Dynamics on the Berkovich Projective Line" ([91], July 2004). Its purpose is to develop the foundations of potential theory and rational dynamics on the Berkovich projective line.

The theory developed here has applications in arithmetic geometry, arithmetic intersection theory, and arithmetic dynamics. In an effort to create a reference which is as useful as possible, we work over an arbitrary complete and algebraically closed non-Archimedean field. We also state our global applications over an arbitrary product formula field whenever possible. Recent work has shown that such generality is essential, even when addressing classical problems over \mathbb{C} . As examples, we note the first author's proof of a Northcott-type finiteness theorem for the dynamical height attached to a nonisotrivial rational function of degree at least 2 over a function field [5] and his joint work with Laura DeMarco [6] on finiteness results for preperiodic points of complex dynamical systems.

We first give a detailed description of the topological structure of the Berkovich projective line. We then introduce the *Hsia kernel*, the fundamental kernel for potential theory (closely related to the Gromov kernel of [47]). Next we define a Laplacian operator on $\mathbb{P}_{\text{Berk}}^1$ and construct theories of capacities, harmonic functions, and subharmonic functions, all strikingly similar to their classical counterparts over \mathbb{C} . We develop a theory of multiplicities for rational maps and give applications to non-Archimedean dynamics, including the construction of a canonical invariant probability measure on $\mathbb{P}_{\text{Berk}}^1$ analogous to the well-known measure on $\mathbb{P}^1(\mathbb{C})$ constructed by Lyubich and by Freire, Lopes, and Mañé. Finally, we investigate Berkovich space analogues of the classical Fatou-Julia theory for rational iteration over \mathbb{C} .

In §7.8, we give an updated treatment (in the special case of \mathbb{P}^1) of the Fekete and Fekete-Szegő theorems from [88], replacing the somewhat esoteric notion of "algebraic capacitability" with the simple notion of compactness. In §7.9, working over an arbitrary product formula field, we prove a generalization of Bilu's equidistribution theorem [24] for algebraic points which are 'small' with respect to the height function attached to a compact Berkovich adelic set. In §10.3, again working over a product formula field, we prove an adelic equidistribution theorem for algebraic points which are 'small' with respect to the dynamical height attached to a rational function of degree at least 2, extending results in [9], [35], and [47].

A more detailed overview of the results in this book can be found in the first author's lecture notes from the 2007 Arizona Winter School [4], and in the Introduction below.

History

This book began as a set of lecture notes from a seminar on the Berkovich projective line held at the University of Georgia during the spring of 2004. The purpose of the seminar was to develop the tools needed to prove an adelic equidistribution theorem for small points with respect to the dynamical height attached to a rational function of degree $d \geq 2$ defined over a number field (Theorem 10.24). Establishing such a theorem had been one of the main goals in our 2002 NSF proposal DMS-0300784.

In [8], the first author and Liang-Chung Hsia had proved an adelic equidistribution theorem for points of $\mathbb{P}^1(\overline{\mathbb{Q}})$ having small dynamical height with respect to the iteration of a polynomial map. Two basic problems remained after that work. First, there was the issue of generalizing the main results of [8] to *rational functions*, rather than just polynomials. It occurs frequently in complex dynamics and potential theory that one needs heavier machinery to deal with rational maps than with polynomials. Second, because the filled Julia set in $\mathbb{P}^1(\mathbb{C}_p)$ of a polynomial over \mathbb{C}_p is often non-compact, the authors of [8] were unable to formulate their result as a true “equidistribution” theorem. Instead, they introduced a somewhat artificial notion of “pseudo-equidistribution” and showed that when the filled Julia set is compact, then pseudo-equidistribution coincides with equidistribution.

The second author, upon learning of the results in [8], suggested that Berkovich's theory might allow those results to be formulated more cleanly. Several years earlier, in [36], he had proposed that Berkovich spaces would be a natural setting for non-Archimedean potential theory.

We thus set out to generalize the results of [8] to a true equidistribution theorem on $\mathbb{P}_{\text{Berk}}^1$, valid for arbitrary rational maps. An important step in this plan was to establish the existence of a canonical invariant measure on $\mathbb{P}_{\text{Berk}}^1$ attached to a rational function of degree at least 2 defined over \mathbb{C}_p , having properties analogous to those of the canonical measure in complex dynamics (see [54, 72]). It was clear that even defining the canonical measure would require significant foundational work.

At roughly the same time, Antoine Chambert-Loir posted a paper to the arXiv preprint server proving (among other things) non-Archimedean Berkovich space analogues of Bilu's equidistribution theorem and the Szpiro-Ullmo-Zhang equidistribution theorem for abelian varieties with good reduction. In the summer of 2003, the first author met with Chambert-Loir in Paris and learned that Chambert-Loir's student Amaury Thuillier had recently defined a Laplacian operator on Berkovich curves. Not knowing exactly what Thuillier had proved, nor when his results might be publicly available, we undertook to develop a measure-valued Laplacian and a theory

of subharmonic functions on $\mathbb{P}_{\text{Berk}}^1$ ourselves, with a view toward applying them in a dynamical setting. The previous year, we had studied Laplacians and their spectral theory on metrized graphs, and that work made it plausible that a Laplacian operator could be constructed on $\mathbb{P}_{\text{Berk}}^1$ by taking an inverse limit of graph Laplacians.

The project succeeded, and we presented our equidistribution theorem at the conference on Arithmetical Dynamical Systems held at CUNY in May 2004. To our surprise, Chambert-Loir, Thuillier, and Pascal Autissier had proved the same theorem using an approach based on Arakelov theory. At the same conference, Rob Benedetto pointed us to the work of Juan Rivera-Letelier, who had independently rediscovered the Berkovich projective line and used it to carry out a deep study of non-Archimedean dynamics. Soon after, we learned that Charles Favre and Rivera-Letelier had independently proved the equidistribution theorem as well.

The realization that three different groups of researchers had been working on similar ideas slowed our plans to develop the theory further. However, over time it became evident that each of the approaches had merit: for example, our proof brought out connections with arithmetic capacities; the proof of Chambert-Loir, Thuillier, and Autissier was later generalized to higher dimensions; and Favre and Rivera-Letelier's proof yielded explicit quantitative error bounds. Ultimately, we, at least, have benefitted greatly from the others' perspectives.

Thus, while this book began as a research monograph, we now view it mainly as an expository work whose goal is to give a systematic presentation of foundational results in potential theory and dynamics on $\mathbb{P}_{\text{Berk}}^1$. Although the approach to potential theory given here is our own, it has overlaps with the theory developed by Thuillier for curves of arbitrary genus. Many of the results in the final two chapters on the dynamics of rational functions were originally discovered by Rivera-Letelier, though some of our proofs are new.

Related works

Amaury Thuillier, in his doctoral thesis [94], established the foundations of potential theory for Berkovich curves of arbitrary genus. Thuillier constructs a Laplacian operator and theories of harmonic and subharmonic functions and gives applications of his work to Arakelov intersection theory. Thuillier's work has great generality and scope, but it is written in a sophisticated language and assumes a considerable amount of machinery. Because this book is written in a more elementary language and deals only with \mathbb{P}^1 , it may be a more accessible introduction to the subject for some readers.

Juan Rivera-Letelier, in his doctoral thesis [81] and subsequent papers [82, 83, 84, 80], has carried out a profound study of the dynamics of rational maps on the Berkovich projective line (though his papers are written in a rather different terminology). Section 10.9 contains an exposition of Rivera-Letelier's work.

Using Rivera-Letelier's ideas, we have simplified and generalized our discussion of multiplicities in Chapter 9 and have greatly extended our original results on the dynamics of rational maps in Chapter 10. It should be noted that Rivera-Letelier's proofs are written with \mathbb{C}_p as the ground field. One of the goals of this book is to establish a reference for parts of his theory which hold over an arbitrary complete and algebraically closed non-Archimedean field.

Charles Favre and Mattias Jonsson [45] have developed a Laplacian operator, and parts of potential theory, in the general context of \mathbb{R} -trees. Their definition of the Laplacian, while ultimately yielding the same operator on $\mathbb{P}_{\text{Berk}}^1$, has a rather different flavor from ours. As noted above, Chambert-Loir [35] and Favre and Rivera-Letelier [46, 47] have given independent proofs of the adelic dynamical equidistribution theorem, as well as constructions of the canonical measure on $\mathbb{P}_{\text{Berk}}^1$ attached to a rational function. Recently Favre and Rivera-Letelier [48] have investigated ergodic theory for rational maps on $\mathbb{P}_{\text{Berk}}^1$. In Section 10.4, we prove a special case of their theorem on the convergence of pullback measures to the canonical measure, which we use as the basis for our development of Fatou-Julia theory.

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Differences from the preliminary version

There are several differences between the present manuscript and the preliminary version [91] posted to the arXiv preprint server in July 2004. For one thing, we have corrected a number of errors in the earlier version.

In addition, we have revised all of the statements and proofs so that they hold over an arbitrary complete, algebraically closed field K endowed with a nontrivial non-Archimedean absolute value, rather than just over the field \mathbb{C}_p . The main difference is that the Berkovich projective line over \mathbb{C}_p is *metrizable* and has *countable* branching at every point, whereas in general the Berkovich projective line over K is nonmetrizable and has uncountable branching. Replacing \mathbb{C}_p by K throughout required a significant reworking of many of our original proofs, since [91] relies in several places on arguments valid for metric spaces but not for an arbitrary compact Hausdorff space. Consequently, the present book makes more demands on the reader in terms of topological prerequisites; for example we now make use of nets rather than sequences in several places.

In some sense, this works against the concrete and “elementary” exposition that we have striven for. However, the changes seem desirable for at least two reasons. First, some proofs become more natural once the crutch of metrizability is removed. Second, and perhaps more importantly, the theory for more general fields is needed for many applications. The first author’s paper [5] is one example of this: it contains a Northcott-type theorem for dynamical canonical heights over a general field k endowed with a product formula; the theorem is proved by working locally at each place v on the Berkovich projective line over \mathbb{C}_v (the smallest complete and algebraically closed field containing k and possessing an absolute value extending the given one $|\cdot|_v$ on k). As another example, Kontsevich and Soibelman [68] have recently used Berkovich’s theory over fields such as the completion of an algebraic closure of $\mathbb{C}((T))$ to study homological mirror symmetry. We mention also the work of Favre and Jonsson [45] on the valuative tree, as well as the related work of Jan Kiwi [66], both of which have applications to complex dynamics.

Here is a summary of the main differences between this work and [91]:

- We have added a detailed Introduction summarizing the work.
- We have added several appendices in order to make the presentation more self-contained.
- We have added a symbol table and an index and updated the bibliography.
- We give a different construction of $\mathbb{P}_{\text{Berk}}^1$ (analogous to the “Proj” construction in algebraic geometry) which makes it easier to understand the action of a rational function.
- We have changed some of our notation and terminology to be compatible with that of the authors mentioned above.

- We have added sections on the Dirichlet pairing and Favre–Rivera-Letelier smoothing.
- We compare our Laplacian with those of Favre, Jonsson, and Rivera-Letelier, and of Thuillier.
- We have included a discussion of \mathbb{R} -trees, and in particular of $\mathbb{P}_{\text{Berk}}^1$ as a “profinite \mathbb{R} -tree”.
- We have expanded our discussion of the Poisson formula on $\mathbb{P}_{\text{Berk}}^1$.
- We have added a section on Thuillier’s short exact sequence describing subharmonic functions in terms of harmonic functions and positive σ -finite measures.
- We have added a section on Hartogs’s lemma, a key ingredient in the work of Favre and Rivera-Letelier.
- We state and prove Berkovich space versions of Bilu’s equidistribution theorem, the dynamical equidistribution theorem for small points, and the arithmetic Fekete-Szegő theorem for \mathbb{P}^1 .
- We have simplified and expanded the discussion in Chapter 9 on analytic multiplicities.
- We have greatly expanded the material on dynamics of rational maps, incorporating the work of Rivera-Letelier and the joint work of Favre and Rivera-Letelier, and including a section on examples.

Introduction

This book has several goals. The first goal is to develop the foundations of potential theory on $\mathbb{P}_{\text{Berk}}^1$, including the definition of a measure-valued Laplacian operator, capacity theory, and a theory of harmonic and subharmonic functions. A second goal is to give applications of potential theory on $\mathbb{P}_{\text{Berk}}^1$, especially to the dynamics of rational maps defined over an arbitrary complete and algebraically closed non-Archimedean field K . A third goal is to provide the reader with a concrete introduction to Berkovich's theory of analytic spaces by focusing on the special case of the Berkovich projective line.

We now outline the contents of the book.

The Berkovich affine and projective lines. Let K be an algebraically closed field which is complete with respect to a nontrivial non-Archimedean absolute value. The topology on K induced by the given absolute value is Hausdorff, but it is also totally disconnected and not locally compact. This makes it difficult to define a good notion of an analytic function on K . Tate dealt with this problem by developing the subject now known as *rigid analysis*, in which one works with a certain Grothendieck topology on K . This leads to a satisfactory theory of analytic functions, but since the underlying topological space is unchanged, difficulties remain for other applications. For example, using only the topology on K , there is no evident way to define a Laplacian operator analogous to the classical Laplacian on \mathbb{C} or to work sensibly with probability measures on K .

However, these difficulties, and many more, can be resolved in a very satisfactory way using Berkovich's theory. The Berkovich affine line $\mathbb{A}_{\text{Berk}}^1$ over K is a locally compact, Hausdorff, and path-connected topological space which contains K (with the topology induced by the given absolute value) as a dense subspace. One obtains the Berkovich projective line $\mathbb{P}_{\text{Berk}}^1$ by adjoining to $\mathbb{A}_{\text{Berk}}^1$ in a suitable manner a point at infinity; the resulting space $\mathbb{P}_{\text{Berk}}^1$ is a compact, Hausdorff, path-connected topological space which contains $\mathbb{P}^1(K)$ (with its natural topology) as a dense subspace. In fact, $\mathbb{A}_{\text{Berk}}^1$ and $\mathbb{P}_{\text{Berk}}^1$ are more than just path-connected: they are *uniquely* path-connected, in the sense that any two distinct points can be joined by a unique arc. The unique path-connectedness is closely related to the fact that $\mathbb{A}_{\text{Berk}}^1$ and $\mathbb{P}_{\text{Berk}}^1$ are endowed with a natural tree structure. (More specifically, they are \mathbb{R} -trees, as defined in §1.4.) The tree structure on $\mathbb{A}_{\text{Berk}}^1$ (resp. $\mathbb{P}_{\text{Berk}}^1$) can

be used to define a *Laplacian operator* in terms of the classical Laplacian on a finite graph. This in turn leads to a theory of harmonic and subharmonic functions which closely parallels the classical theory over \mathbb{C} .

The definition of $\mathbb{A}_{\text{Berk}}^1$ is quite simple and makes sense with K replaced by an arbitrary field k endowed with a (possibly Archimedean or even trivial) absolute value. As a set, $\mathbb{A}_{\text{Berk},k}^1$ consists of all multiplicative seminorms on the polynomial ring $k[T]$ which extend the usual absolute value on k . (A *multiplicative seminorm* on a ring A is a function $[\]_x : A \rightarrow \mathbb{R}_{\geq 0}$ satisfying $[0]_x = 0$, $[1]_x = 1$, $[fg]_x = [f]_x \cdot [g]_x$, and $[f+g]_x \leq [f]_x + [g]_x$ for all $f, g \in A$.) By an aesthetically desirable abuse of notation, we will identify seminorms $[\]_x$ with points $x \in \mathbb{A}_{\text{Berk},k}^1$, and we will usually omit explicit reference to the field k , writing simply $\mathbb{A}_{\text{Berk}}^1$. The topology on $\mathbb{A}_{\text{Berk},k}^1$ is the weakest one for which $x \mapsto [f]_x$ is continuous for every $f \in k[T]$.

To motivate this definition, we observe that in the classical setting, every multiplicative seminorm on $\mathbb{C}[T]$ which extends the usual absolute value on \mathbb{C} is of the form $f \mapsto |f(z)|$ for some $z \in \mathbb{C}$. (This can be deduced from the well-known Gelfand-Mazur theorem from functional analysis.) It is then easy to see that $\mathbb{A}_{\text{Berk},\mathbb{C}}^1$ is homeomorphic to \mathbb{C} itself and also to the Gelfand spectrum (i.e., the space of all maximal ideals) of $\mathbb{C}[T]$.

In the non-Archimedean world, K can once again be identified with the Gelfand space of maximal ideals in $K[T]$, but now there are many more multiplicative seminorms on $K[T]$ than just the ones given by evaluation at a point of K . The prototypical example arises by fixing a closed disc $D(a, r) = \{z \in K : |z - a| \leq r\}$ in K and defining $[\]_{D(a,r)}$ by

$$[f]_{D(a,r)} = \sup_{z \in D(a,r)} |f(z)|.$$

It is an elementary consequence of Gauss's lemma that $[\]_{D(a,r)}$ is *multiplicative*, and the other axioms for a seminorm are trivially satisfied. Thus each disc $D(a, r)$ gives rise to a point of $\mathbb{A}_{\text{Berk}}^1$. Note that this includes discs for which $r \notin |K^\times|$, i.e., “irrational discs” for which the set $\{z \in K : |z - a| = r\}$ is empty. We may consider the point a as a “degenerate” disc of radius zero. (If $r > 0$, then $[\]_{D(a,r)}$ is not only a seminorm, but a norm.) It is not hard to see that distinct discs $D(a, r)$ with $r \geq 0$ give rise to distinct multiplicative seminorms on $K[T]$, and therefore the set of all such discs embeds naturally into $\mathbb{A}_{\text{Berk}}^1$.

Suppose $x, x' \in \mathbb{A}_{\text{Berk}}^1$ are distinct points corresponding to the (possibly degenerate) discs $D(a, r), D(a', r')$, respectively. The unique path in $\mathbb{A}_{\text{Berk}}^1$ between x and x' has a very intuitive description. If $D(a, r) \subset D(a', r')$, it consists of all points of $\mathbb{A}_{\text{Berk}}^1$ corresponding to discs containing $D(a, r)$ and contained in $D(a', r')$. The set of all such “intermediate discs” is totally ordered by containment, and if $a = a'$, it is just $\{D(a, t) : r \leq t \leq r'\}$. If $D(a, r)$ and $D(a', r')$ are disjoint, the unique path between x and x' consists of all points of $\mathbb{A}_{\text{Berk}}^1$ corresponding to discs which are either of the form $D(a, t)$ with $r \leq t \leq |a - a'|$ or of the form $D(a', t)$ with $r' \leq t \leq |a - a'|$.

The disc $D(a, |a - a'|) = D(a', |a - a'|)$ is the smallest one containing both $D(a, r)$ and $D(a', r')$, and if $x \vee x'$ denotes the point of $\mathbb{A}_{\text{Berk}}^1$ corresponding to $D(a, |a - a'|)$, then the path from x to x' is just the path from x to $x \vee x'$ followed by the path from $x \vee x'$ to x' .

In particular, if a, a' are distinct points of K , one can visualize the path in $\mathbb{A}_{\text{Berk}}^1$ from a to a' as follows: increase the “radius” of the degenerate disc $D(a, 0)$ until a disc $D(a, r)$ is reached which also contains a' . This disc can also be written as $D(a', s)$ with $s = |a - a'|$. Now decrease s until the radius reaches zero. This “connects” the totally disconnected space K by adding points corresponding to closed discs in K . In order to obtain a *compact* space, however, it is necessary in general to add even more points, for K may not be *spherically complete* (this happens, e.g., when $K = \mathbb{C}_p$): there may be decreasing sequences of closed discs with empty intersection. Intuitively, we need to add in points corresponding to such sequences in order to obtain a space which has a chance of being compact. More precisely, returning to the definition of $\mathbb{A}_{\text{Berk}}^1$ in terms of multiplicative seminorms, if $\{D(a_i, r_i)\}$ is any decreasing nested sequence of closed discs, then the map

$$f \mapsto \lim_{i \rightarrow \infty} [f]_{D(a_i, r_i)}$$

defines a multiplicative seminorm on $K[T]$ extending the usual absolute value on K . One can show that two sequences of discs with empty intersection define the same seminorm if and only if the sequences are *cofinal*. This yields a large number of additional points of $\mathbb{A}_{\text{Berk}}^1$. According to *Berkovich’s classification theorem*, we have now described all the points of $\mathbb{A}_{\text{Berk}}^1$: each point $x \in \mathbb{A}_{\text{Berk}}^1$ corresponds to a decreasing nested sequence $\{D(a_i, r_i)\}$ of closed discs, and we can categorize the points of $\mathbb{A}_{\text{Berk}}^1$ into four types according to the nature of $D = \bigcap D(a_i, r_i)$:

- (I) D is a point of K .
- (II) D is a closed disc with radius belonging to $|K^\times|$.
- (III) D is a closed disc with radius *not* belonging to $|K^\times|$.
- (IV) $D = \emptyset$.

As a set, $\mathbb{P}_{\text{Berk}}^1$ can be obtained from $\mathbb{A}_{\text{Berk}}^1$ by adding a type I point denoted ∞ . The topology on $\mathbb{P}_{\text{Berk}}^1$ is that of the one-point compactification.

Following Rivera-Letelier, we write \mathbb{H}_{Berk} for the subset of $\mathbb{P}_{\text{Berk}}^1$ consisting of all points of type II, III, or IV (Berkovich “hyperbolic space”). Note that \mathbb{H}_{Berk} consists of precisely the points in $\mathbb{P}_{\text{Berk}}^1$ for which $[\]_x$ is a norm. We also write $\mathbb{H}_{\text{Berk}}^{\mathbb{Q}}$ for the set of type II points and $\mathbb{H}_{\text{Berk}}^{\mathbb{R}}$ for the set of points of type II or III.

The description of points of $\mathbb{A}_{\text{Berk}}^1$ in terms of closed discs is very useful, because it allows one to visualize quite concretely the abstract space of multiplicative seminorms which we started with. It also allows us to understand in a more concrete way the natural partial order on $\mathbb{A}_{\text{Berk}}^1$ in which $x \leq y$ if and only if $[f]_x \leq [f]_y$ for all $f \in K[T]$. In terms of discs, if x, y are points of type I, II, or III, one can show that $x \leq y$ if and only if the disc

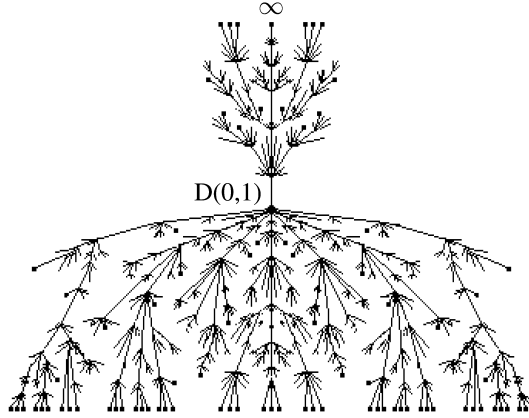


FIGURE 0.1. The Berkovich Projective Line

corresponding to x is contained in the disc corresponding to y . (We leave it as an exercise to the reader to extend this description of the partial order to points of type IV.) For any pair of points $x, y \in \mathbb{A}_{\text{Berk}}^1$, there is a unique least upper bound $x \vee y \in \mathbb{A}_{\text{Berk}}^1$ with respect to this partial order. We can extend the partial order to $\mathbb{P}_{\text{Berk}}^1$ by declaring that $x \leq \infty$ for all $x \in \mathbb{A}_{\text{Berk}}^1$.

Writing

$$[x, x'] = \{z \in \mathbb{P}_{\text{Berk}}^1 : x \leq z \leq x'\} \cup \{z \in \mathbb{P}_{\text{Berk}}^1 : x' \leq z \leq x\},$$

it is easy to see that the unique path between $x, y \in \mathbb{P}_{\text{Berk}}^1$ is just

$$[x, x \vee y] \cup [x \vee y, y].$$

There is a canonical metric ρ on \mathbb{H}_{Berk} which is of great importance for potential theory. To define it, we first define a function $\text{diam} : \mathbb{A}_{\text{Berk}}^1 \rightarrow \mathbb{R}_{\geq 0}$ by setting $\text{diam}(x) = \lim r_i$ if x corresponds to the nested sequence $\{D(a_i, r_i)\}$. This is easily checked to be well-defined, independent of the choice of nested sequence. If $x \in \mathbb{H}_{\text{Berk}}^{\mathbb{R}}$, then $\text{diam}(x)$ is just the diameter (= radius) of the corresponding closed disc. Because K is complete, if x is of type IV, then $\text{diam}(x) > 0$. Thus $\text{diam}(x) = 0$ for $x \in \mathbb{A}_{\text{Berk}}^1$ of type I, and $\text{diam}(x) > 0$ for $x \in \mathbb{H}_{\text{Berk}}$.

If $x, y \in \mathbb{H}_{\text{Berk}}$ with $x \leq y$, we define

$$\rho(x, y) = \log_v \frac{\text{diam}(y)}{\text{diam}(x)},$$

where \log_v denotes the logarithm to the base q_v , with $q_v > 1$ a fixed real number chosen so that $-\log_v |\cdot|$ is a prescribed normalized valuation on K .

More generally, for $x, y \in \mathbb{H}_{\text{Berk}}$ arbitrary, we define the *path distance metric* $\rho(x, y)$ by

$$\rho(x, y) = \rho(x, x \vee y) + \rho(y, x \vee y).$$

It is not hard to verify that ρ defines a metric on \mathbb{H}_{Berk} . One can extend ρ to a singular function on $\mathbb{P}_{\text{Berk}}^1$ by declaring that if $x \in \mathbb{P}^1(K)$ and $y \in \mathbb{P}_{\text{Berk}}^1$,

we have $\rho(x, y) = \infty$ if $x \neq y$ and 0 if $x = y$. However, we usually only consider ρ as being defined on \mathbb{H}_{Berk} .

It is important to note that the topology on \mathbb{H}_{Berk} defined by the metric ρ is *not* the subspace topology induced from the Berkovich (or Gelfand) topology on $\mathbb{P}_{\text{Berk}}^1 \supset \mathbb{H}_{\text{Berk}}$; it is strictly finer than the subspace topology.

The group $\text{PGL}(2, K)$ of *Möbius transformations* acts continuously on $\mathbb{P}_{\text{Berk}}^1$ in a natural way compatible with the usual action on $\mathbb{P}^1(K)$, and this action preserves \mathbb{H}_{Berk} , $\mathbb{H}_{\text{Berk}}^{\mathbb{Q}}$, and $\mathbb{H}_{\text{Berk}}^{\mathbb{R}}$. Using the definition of $\mathbb{P}_{\text{Berk}}^1$ in terms of multiplicative seminorms (and extending each $[\]_x$ to a seminorm on its local ring in the quotient field $K(T)$), we have $[f]_{M(x)} = [f \circ M]_x$ for each $M \in \text{PGL}(2, K)$. The action of $\text{PGL}(2, K)$ on $\mathbb{P}_{\text{Berk}}^1$ can also be described concretely in terms of Berkovich's classification theorem, using the fact that each $M \in \text{PGL}(2, K)$ takes closed discs to closed discs. An important observation is that $\text{PGL}(2, K)$ acts *isometrically* on \mathbb{H}_{Berk} , i.e.,

$$\rho(M(x), M(y)) = \rho(x, y)$$

for all $x, y \in \mathbb{H}_{\text{Berk}}$ and all $M \in \text{PGL}(2, K)$. This shows that the path distance metric ρ is “coordinate-free”.

The diameter function diam can also be used to extend the usual distance function $|x - y|$ on K to $\mathbb{A}_{\text{Berk}}^1$. We call this extension the *Hsia kernel* and denote it by $\delta(x, y)_{\infty}$. Formally, for $x, y \in \mathbb{A}_{\text{Berk}}^1$ we have

$$\delta(x, y)_{\infty} = \text{diam}(x \vee y) .$$

It is easy to see that if $x, y \in K$, then $\delta(x, y)_{\infty} = |x - y|$. More generally, one has the formula

$$\delta(x, y)_{\infty} = \limsup_{(x_0, y_0) \rightarrow (x, y)} |x_0 - y_0| ,$$

where $(x_0, y_0) \in K \times K$ and the convergence implicit in the $\lim \sup$ is with respect to the product topology on $\mathbb{P}_{\text{Berk}}^1 \times \mathbb{P}_{\text{Berk}}^1$. The Hsia kernel satisfies all of the axioms for an ultrametric with one exception: we have $\delta(x, x)_{\infty} > 0$ for $x \in \mathbb{H}_{\text{Berk}}$.

The function $-\log_v \delta(x, y)_{\infty}$, which generalizes the usual potential theory kernel $-\log_v |x - y|$, leads to a theory of capacities on $\mathbb{P}_{\text{Berk}}^1$ which generalizes that of [88] and which has many features in common with classical capacity theory over \mathbb{C} .

There is also a *generalized Hsia kernel* $\delta(x, y)_{\zeta}$ with respect to an arbitrary point $\zeta \in \mathbb{P}_{\text{Berk}}^1$; we refer the reader to §4.4 for details.

We now come to an important description of $\mathbb{P}_{\text{Berk}}^1$ as a *profinite \mathbb{R} -tree*. An \mathbb{R} -tree is a metric space (T, d) such that for each distinct pair of points $x, y \in T$, there is a unique arc in T from x to y , and this arc is a geodesic. (See Appendix B for a more detailed discussion of \mathbb{R} -trees.) A *branch point* is a point $x \in T$ for which $T \setminus \{x\}$ has either one or more than two connected components. A *finite \mathbb{R} -tree* is an \mathbb{R} -tree which is compact and has only finitely many branch points. Intuitively, a finite \mathbb{R} -tree is just a finite tree

in the usual graph-theoretic sense, but where the edges are thought of as line segments having definite lengths. Finally, a *profinite* \mathbb{R} -tree is an inverse limit of finite \mathbb{R} -trees.

Let us consider how these definitions play out for $\mathbb{P}_{\text{Berk}}^1$. If $S \subset \mathbb{P}_{\text{Berk}}^1$, define the *convex hull* of S to be the smallest path-connected subset of $\mathbb{P}_{\text{Berk}}^1$ containing S . (This is the same as the set of all paths between points of S .) By abuse of terminology, a *finite subgraph* of $\mathbb{P}_{\text{Berk}}^1$ will mean the convex hull of a finite subset $S \subset \mathbb{H}_{\text{Berk}}^{\mathbb{R}}$. Every finite subgraph Γ , when endowed with the induced path distance metric ρ , is a finite \mathbb{R} -tree, and the collection of all finite subgraphs of $\mathbb{P}_{\text{Berk}}^1$ is a directed set under inclusion. Moreover, if $\Gamma \leq \Gamma'$, then by a basic property of \mathbb{R} -trees, there is a continuous *retraction map* $r_{\Gamma', \Gamma} : \Gamma' \rightarrow \Gamma$. In §1.4, we will show that $\mathbb{P}_{\text{Berk}}^1$ is homeomorphic to the inverse limit $\varprojlim \Gamma$ over all finite subgraphs $\Gamma \subset \mathbb{P}_{\text{Berk}}^1$. (Intuitively, this is just a topological formulation of Berkovich's classification theorem.) This description of $\mathbb{P}_{\text{Berk}}^1$ as a profinite \mathbb{R} -tree provides a convenient way to visualize the topology on $\mathbb{P}_{\text{Berk}}^1$: two points are “close” if they retract to the same point of a “large” finite subgraph. For each Γ , we let $r_{\mathbb{P}_{\text{Berk}}^1, \Gamma}$ be the natural retraction map from $\mathbb{P}_{\text{Berk}}^1$ to Γ coming from the universal property of the inverse limit.

A fundamental system of open neighborhoods for the topology on $\mathbb{P}_{\text{Berk}}^1$ is given by the *open affinoid subsets*, which are the sets of the form $r_{\mathbb{P}_{\text{Berk}}^1, \Gamma}^{-1}(V)$ for Γ a finite subgraph of $\mathbb{H}_{\text{Berk}}^{\mathbb{R}}$ and V an open subset of Γ . We will refer to a connected open affinoid subset of $\mathbb{P}_{\text{Berk}}^1$ as a *simple domain*. Simple domains can be completely characterized as the connected open subsets of $\mathbb{P}_{\text{Berk}}^1$ having a finite (nonzero) number of boundary points, all of which are contained in $\mathbb{H}_{\text{Berk}}^{\mathbb{R}}$. If U is an open subset of $\mathbb{P}_{\text{Berk}}^1$, a *simple subdomain* of U is defined to be a simple domain whose closure is contained in U .

Laplacians. The profinite \mathbb{R} -tree structure on $\mathbb{P}_{\text{Berk}}^1$ leads directly to the construction of a Laplacian operator. On a finite subgraph Γ of $\mathbb{P}_{\text{Berk}}^1$ (or, more generally, on any ‘metrized graph’; see Chapter 3 for details), there is a natural Laplacian operator Δ_{Γ} generalizing the well-known combinatorial Laplacian on a weighted graph. If $f : \Gamma \rightarrow \mathbb{R}$ is continuous, and \mathcal{C}^2 except at a finite number of points, then there is a unique Borel measure $\Delta_{\Gamma}(f)$ of total mass zero on Γ such that

$$(0.1) \quad \int_{\Gamma} \psi \Delta_{\Gamma}(f) = \int_{\Gamma} f'(x) \psi'(x) dx$$

for all continuous, piecewise affine functions ψ on Γ . The measure $\Delta_{\Gamma}(f)$ has a discrete part and a continuous part. At each $P \in \Gamma$ which is either a branch point of Γ or a point where $f(x)$ fails to be \mathcal{C}^2 , $\Delta_{\Gamma}(f)$ has a point mass equal to the negative of the sum of the directional derivatives of $f(x)$ on the edges emanating from P . On the intervening edges, it is given by $-f''(x)dx$. (See Chapter 3 for details.)

We define $\text{BDV}(\Gamma)$ to be the space of all continuous real-valued functions f on Γ for which the distribution defined by

$$(0.2) \quad \psi \mapsto \int_{\Gamma} f \Delta_{\Gamma}(\psi) ,$$

for all ψ as above, is represented by a bounded signed Borel measure $\Delta_{\Gamma}(f)$. A simple integration by parts argument shows that this measure coincides with the one defined by (0.1) when f is sufficiently smooth. The name “BDV” is an abbreviation for “Bounded Differential Variation”. We call the measure $\Delta_{\Gamma}(f)$ the *Laplacian* of f on Γ .

The Laplacian satisfies an important compatibility property with respect to the partial order on the set of finite subgraphs of $\mathbb{P}_{\text{Berk}}^1$ given by containment: if $\Gamma \leq \Gamma'$ and $f \in \text{BDV}(\Gamma')$, then

$$(0.3) \quad \Delta_{\Gamma}(f|_{\Gamma}) = (r_{\Gamma',\Gamma})_* \Delta_{\Gamma'}(f) .$$

We define $\text{BDV}(\mathbb{P}_{\text{Berk}}^1)$ to be the collection of all functions $f : \mathbb{P}_{\text{Berk}}^1 \rightarrow \mathbb{R} \cup \{\pm\infty\}$ such that:

- $f|_{\Gamma} \in \text{BDV}(\Gamma)$ for each finite subgraph Γ .
- The measures $|\Delta_{\Gamma}(f)|$ have uniformly bounded total mass.

Note that belonging to $\text{BDV}(\mathbb{P}_{\text{Berk}}^1)$ imposes no condition on the values of f at points of $\mathbb{P}^1(K)$.

Using the compatibility property (0.3), one shows that if $f \in \text{BDV}(\mathbb{P}_{\text{Berk}}^1)$, then the collection of measures $\{\Delta_{\Gamma}\}$ “cohere” to give a unique Borel measure $\Delta(f)$ of total mass zero on the inverse limit space $\mathbb{P}_{\text{Berk}}^1$ satisfying

$$(r_{\mathbb{P}_{\text{Berk}}^1, \Gamma})_* \Delta(f) = \Delta_{\Gamma}(f)$$

for all finite subgraphs Γ of $\mathbb{P}_{\text{Berk}}^1$. We call $\Delta(f)$ the *Laplacian* of f on $\mathbb{P}_{\text{Berk}}^1$.

Similarly, if U is a domain (i.e., a nonempty connected open subset) in $\mathbb{P}_{\text{Berk}}^1$, one defines a class $\text{BDV}(U)$ of functions $f : U \rightarrow \mathbb{R} \cup \{\pm\infty\}$ for which the Laplacian $\Delta_{\overline{U}}(f)$ is a bounded Borel measure of total mass zero supported on the closure of U . The measure $\Delta_{\overline{U}}(f)$ has the property that

$$(r_{\overline{U}, \Gamma})_* \Delta(f) = \Delta_{\Gamma}(f)$$

for all finite subgraphs Γ of $\mathbb{P}_{\text{Berk}}^1$ contained in U .

As a concrete example, fix $y \in \mathbb{A}_{\text{Berk}}^1$ and let $f : \mathbb{P}_{\text{Berk}}^1 \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be defined by $f(\infty) = -\infty$ and

$$f(x) = -\log_v \delta(x, y)_{\infty}$$

for $x \in \mathbb{A}_{\text{Berk}}^1$. Then $f \in \text{BDV}(\mathbb{P}_{\text{Berk}}^1)$, and

$$(0.4) \quad \Delta f = \delta_y - \delta_{\infty}$$

is a discrete measure on $\mathbb{P}_{\text{Berk}}^1$ supported on $\{y, \infty\}$. Intuitively, the explanation for the formula (0.4) is as follows. The function f is locally constant away from the path $\Lambda = [y, \infty]$ from y to ∞ ; more precisely, we have $f(x) = f(r_{\mathbb{P}_{\text{Berk}}^1, \Lambda}(x))$. Moreover, the restriction of f to Λ is linear (with

respect to the distance function ρ) with slope -1 . For every suitable test function ψ , we therefore have the “heuristic” calculation

$$\int_{\mathbb{P}_{\text{Berk}}^1} \psi \Delta f = \int_{\Lambda} f'(x) \psi'(x) dx = - \int_y^{\infty} \psi'(x) dx = \psi(y) - \psi(\infty) .$$

(To make this calculation rigorous, one needs to exhaust $\Lambda = [y, \infty]$ by an increasing sequence of line segments $\Gamma \subset \mathbb{H}_{\text{Berk}}^{\mathbb{R}}$ and then observe that the corresponding measures $\Delta_{\Gamma} f$ converge weakly to $\delta_y - \delta_{\infty}$.)

Equation (0.4) shows that $-\log_v \delta(x, y)_{\infty}$, like its classical counterpart $-\log |x - y|$ over \mathbb{C} , is a fundamental solution (in the sense of distributions) to the Laplace equation. This “explains” why $-\log_v \delta(x, y)_{\infty}$ is the correct kernel for doing potential theory.

More generally, let $\varphi \in K(T)$ be a nonzero rational function with zeros and poles given by the divisor $\text{div}(\varphi)$ on $\mathbb{P}^1(K)$. The usual action of φ on $\mathbb{P}^1(K)$ extends naturally to an action of φ on $\mathbb{P}_{\text{Berk}}^1$, and there is a continuous function $-\log_v[\varphi]_x : \mathbb{P}_{\text{Berk}}^1 \rightarrow \mathbb{R} \cup \{\pm\infty\}$ extending the usual map $x \mapsto -\log_v |\varphi(x)|$ on $\mathbb{P}^1(K)$. One derives from (0.4) the following version of the *Poincaré-Lelong formula*:

$$\Delta_{\mathbb{P}_{\text{Berk}}^1} (-\log_v[\varphi]_x) = \delta_{\text{div}(\varphi)} .$$

Capacities. Fix $\zeta \in \mathbb{P}_{\text{Berk}}^1$, and let E be a compact subset of $\mathbb{P}_{\text{Berk}}^1 \setminus \{\zeta\}$. (For concreteness, the reader may wish to imagine that $\zeta = \infty$.) By analogy with the classical theory over \mathbb{C} and also with the non-Archimedean theory developed in [88], one can define the *logarithmic capacity* of E with respect to ζ . This is done as follows.

Given a probability measure ν on $\mathbb{P}_{\text{Berk}}^1$ with support contained in E , we define the energy integral

$$I_{\zeta}(\nu) = \iint_{E \times E} -\log_v \delta(x, y)_{\zeta} d\nu(x) d\nu(y) .$$

Letting ν vary over the collection $\mathbb{P}(E)$ of all probability measures supported on E , one defines the *Robin constant*

$$V_{\zeta}(E) = \inf_{\nu \in \mathbb{P}(E)} I_{\zeta}(\nu) .$$

The *logarithmic capacity* of E relative to ζ is then defined to be

$$\gamma_{\zeta}(E) = q_v^{-V_{\zeta}(E)} .$$

For an arbitrary set H , the logarithmic capacity $\gamma_{\zeta}(H)$ is defined by

$$\gamma_{\zeta}(H) = \sup_{\text{compact } E \subset H} \gamma_{\zeta}(E) .$$

A countably supported probability measure must have point masses, and $\delta(x, x)_{\zeta} = 0$ for $x \in \mathbb{P}^1(K) \setminus \{\zeta\}$; thus $V_{\zeta}(E) = +\infty$ when $E \subset \mathbb{P}^1(K)$ is countable, so every countable subset of $\mathbb{P}^1(K)$ has capacity zero. On the other hand, for a “nonclassical” point $x \in \mathbb{H}_{\text{Berk}}$ we have $V_{\zeta}(\{x\}) < +\infty$,

since $\delta(x, x)_\zeta > 0$, and therefore $\gamma_\zeta(\{x\}) > 0$. In particular, a singleton set can have positive capacity, a phenomenon which has no classical analogue. More generally, if $E \cap \mathbb{H}_{\text{Berk}} \neq \emptyset$, then $\gamma_\zeta(E) > 0$.

As a more elaborate example, if $K = \mathbb{C}_p$ and $E = \mathbb{Z}_p \subset \mathbb{A}^1(\mathbb{C}_p) \subset \mathbb{A}_{\text{Berk}, \mathbb{C}_p}^1$, then $\gamma_\infty(E) = p^{-1/(p-1)}$. Since $\delta(x, y)_\infty = |x - y|$ for $x, y \in K$, this follows from the same computation as in [88, Example 4.1.24].

For fixed E , the property of E having capacity 0 relative to ζ is independent of the point $\zeta \notin E$.

If E is compact and $\gamma_\zeta(E) > 0$, we show that there is a *unique* probability measure $\mu_{E, \zeta}$ on E , called the *equilibrium measure* of E with respect to ζ , which minimizes energy (i.e., for which $I_\zeta(\mu_{E, \zeta}) = V_\zeta(E)$). As in the classical case, $\mu_{E, \zeta}$ is always supported on the boundary of E .

Closely linked to the theory of capacities is the theory of *potential functions*. For each probability measure ν supported on $\mathbb{P}_{\text{Berk}}^1 \setminus \{\zeta\}$, one defines the potential function $u_\nu(z, \zeta)$ by

$$u_\nu(z, \zeta) = \int -\log_v \delta(z, w)_\zeta d\nu(w) .$$

As in classical potential theory, potential functions need not be continuous, but they do share several of the distinguishing features of continuous functions. For example, $u_\nu(z, \zeta)$ is lower semicontinuous, and it is continuous at each $z \notin \text{supp}(\nu)$. Potential functions on $\mathbb{P}_{\text{Berk}}^1$ satisfy the following analogues of Maria's theorem and Frostman's theorem from complex potential theory:

THEOREM (Maria). *If $u_\nu(z, \zeta) \leq M$ on $\text{supp}(\nu)$, then $u_\nu(z, \zeta) \leq M$ for all $z \in \mathbb{P}_{\text{Berk}}^1 \setminus \{\zeta\}$.*

THEOREM (Frostman). *If a compact set E has positive capacity, then the equilibrium potential $u_E(z, \zeta)$ satisfies $u_E(z, \zeta) \leq V_\zeta(E)$ for all $z \in \mathbb{P}_{\text{Berk}}^1 \setminus \{\zeta\}$, and $u_E(z, \zeta) = V_\zeta(E)$ for all $z \in E$ outside a set of capacity zero.*

As in capacity theory over \mathbb{C} , one can also define the *transfinite diameter* and the *Chebyshev constant* of E , and they both turn out to be equal to the logarithmic capacity of E . (In fact, we define three different variants of the Chebyshev constant and prove that they are all equal.)

As an arithmetic application of the theory of capacities on $\mathbb{P}_{\text{Berk}}^1$, we formulate generalizations to $\mathbb{P}_{\text{Berk}}^1$ of the Fekete and Fekete-Szegő theorems from [88]. The proofs are easy, since they go by reducing the general case to the special case of RL-domains, which was already treated in [88]. Nonetheless, the results are aesthetically pleasing because in their statement, the simple notion of compactness replaces the awkward concept of “algebraic capacity”. The possibility for such a reformulation is directly related to the fact that $\mathbb{P}_{\text{Berk}}^1$ is compact, while $\mathbb{P}^1(K)$ is not.

Harmonic functions. If U is a domain in $\mathbb{P}_{\text{Berk}}^1$, a real-valued function $f : U \rightarrow \mathbb{R}$ is called *strongly harmonic* on U if it is continuous, belongs to $\text{BDV}(U)$, and if $\Delta_{\overline{U}}(f)$ is supported on ∂U . The function f is *harmonic* on U if every point $x \in U$ has a connected open neighborhood on which f is strongly harmonic.

Harmonic functions on domains $U \subseteq \mathbb{P}_{\text{Berk}}^1$ satisfy many properties analogous to their classical counterparts over \mathbb{C} . For example, a harmonic function which attains its maximum or minimum value on U must be constant. There is also an analogue of the Poisson formula: if f is a harmonic function on an open affinoid U , then f extends uniquely to the boundary ∂U , and the values of f on U can be computed explicitly in terms of $f|_{\partial U}$. A version of Harnack's principle holds as well: the limit of a monotonically increasing sequence of nonnegative harmonic functions on U is either harmonic or identically $+\infty$. Even better than the classical case (where a hypothesis of uniform convergence is required), a *pointwise* limit of harmonic functions is automatically harmonic. As is the case over \mathbb{C} , harmonicity is preserved under pullbacks by meromorphic functions.

Fix $\zeta \in \mathbb{P}_{\text{Berk}}^1$, and let E be a compact subset of $\mathbb{P}_{\text{Berk}}^1 \setminus \{\zeta\}$. We define the *Green's function of E relative to ζ* to be

$$G(z, \zeta; E) = V_{\zeta}(E) - u_E(z, \zeta)$$

for all $z \in \mathbb{P}_{\text{Berk}}^1$. We show that the Green's function is everywhere non-negative and that it is strictly positive on the connected component U_{ζ} of $\mathbb{P}_{\text{Berk}}^1 \setminus E$ containing ζ . Also, $G(z, \zeta; E)$ is finite on $\mathbb{P}_{\text{Berk}}^1 \setminus \{\zeta\}$, with a logarithmic singularity at ζ , and it is harmonic on $U_{\zeta} \setminus \{\zeta\}$. Additionally, $G(z, \zeta; E)$ is identically zero on the complement of U_{ζ} outside a set of capacity zero. The Laplacian of $G(z, \zeta; E)$ on $\mathbb{P}_{\text{Berk}}^1$ is equal to $\delta_{\zeta} - \mu_{E, \zeta}$. As in the classical case, the Green's function is symmetric as a function of z and ζ : we have

$$G(z_1, z_2; E) = G(z_2, z_1; E)$$

for all $z_1, z_2 \notin E$. In a satisfying improvement over the theory for $\mathbb{P}^1(\mathbb{C}_p)$ in [88], the role of $G(z, \zeta; E)$ as a reproducing kernel for the Berkovich space Laplacian becomes evident.

As an arithmetic application of the theory of Green's functions and capacities on $\mathbb{P}_{\text{Berk}}^1$, we prove a Berkovich space generalization of Bilu's equidistribution theorem for a rather general class of adelic heights.

Subharmonic functions. We give two characterizations of what it means for a function on a domain $U \subseteq \mathbb{P}_{\text{Berk}}^1$ to be subharmonic. The first, which we take as the definition, is as follows. We say that a function $f : U \rightarrow \mathbb{R} \cup \{-\infty\}$ is *strongly subharmonic* if it is upper semicontinuous, satisfies a further technical semicontinuity hypothesis at points of $\mathbb{P}^1(K)$, and if the positive part of $\Delta_{\overline{U}}(f)$ is supported on ∂U . We say that f is *subharmonic* on U if every point of U has a connected open neighborhood on which f is strongly subharmonic. We also say that f is *superharmonic* on U if $-f$ is subharmonic on U . As an example, if ν is a probability

measure on $\mathbb{P}_{\text{Berk}}^1$ and $\zeta \notin \text{supp}(\nu)$, the potential function $u_\nu(x, \zeta)$ is strongly superharmonic on $\mathbb{P}_{\text{Berk}}^1 \setminus \{\zeta\}$ and is strongly subharmonic on $\mathbb{P}_{\text{Berk}}^1 \setminus \text{supp}(\nu)$. A function f is harmonic on U if and only if it is both subharmonic and superharmonic on U .

As a second characterization of subharmonic functions, we say that $f : U \rightarrow \mathbb{R} \cup \{-\infty\}$ (not identically $-\infty$) is *domination subharmonic* on the domain U if it is upper semicontinuous and if for each simple subdomain V of U and each harmonic function h on V for which $f \leq h$ on ∂V , we have $f \leq h$ on V . A fundamental fact, proved in §8.2, is that f is subharmonic on U if and only if it is domination subharmonic on U .

Like harmonic functions, subharmonic functions satisfy the Maximum Principle: if U is a domain in $\mathbb{P}_{\text{Berk}}^1$ and f is a subharmonic function which attains its maximum value on U , then f is constant. In addition, subharmonic functions on domains in $\mathbb{P}_{\text{Berk}}^1$ are stable under many of the same operations (e.g., convex combinations, maximum, monotone convergence, uniform convergence) as their classical counterparts. There is also an analogue of the Riesz Decomposition Theorem, according to which a subharmonic function on a simple subdomain $V \subset U$ can be written as the difference of a harmonic function and a potential function. We also show that subharmonic functions can be well-approximated by continuous functions of a special form, which we call *smooth* functions.

In §8.10, we define the notion of an Arakelov-Green's function on $\mathbb{P}_{\text{Berk}}^1$ and establish an *energy minimization principle* used in the proof of the main result in [9]. We give two proofs of the Energy Minimization Principle, one using the theory of subharmonic functions and another using the Dirichlet pairing.

Multiplicities. If $\varphi \in K(T)$ is a nonconstant rational function, then as discussed above, the action of φ on $\mathbb{P}^1(K)$ extends naturally to an action of φ on $\mathbb{P}_{\text{Berk}}^1$. We use the theory of Laplacians to give an analytic construction of *multiplicities* for points in $\mathbb{P}_{\text{Berk}}^1$ which generalize the usual multiplicity of φ at a point $a \in \mathbb{P}^1(K)$ (i.e., the multiplicity of a as a preimage of $b = \varphi(a) \in \mathbb{P}^1(K)$). Using the theory of multiplicities, we show that the extended map $\varphi : \mathbb{P}_{\text{Berk}}^1 \rightarrow \mathbb{P}_{\text{Berk}}^1$ is a surjective open mapping. We also obtain a purely topological interpretation of multiplicities, which shows that our multiplicities coincide with those defined by Rivera-Letelier. For each $a \in \mathbb{P}_{\text{Berk}}^1$, the multiplicity of φ at a is a positive integer, and if $\text{char}(K) = 0$, it is equal to 1 if and only if φ is locally injective at a . For each $b \in \mathbb{P}_{\text{Berk}}^1$, the sum of the multiplicities of φ over all preimages of b is equal to the degree of φ .

Using these multiplicities, we define the pushforward and pullback of a bounded Borel measure on $\mathbb{P}_{\text{Berk}}^1$ under φ . The pushforward and pullback measures satisfy the expected functoriality properties; for example, if f is subharmonic on U , then $f \circ \varphi$ is subharmonic on $\varphi^{-1}(U)$ and the Laplacian of $f \circ \varphi$ is the pullback under φ of the Laplacian of f .

Applications to the dynamics of rational maps. Though Berkovich introduced his theory of analytic spaces with rather different goals in mind, Berkovich spaces are well adapted to the study of non-Archimedean dynamics. The fact that the topological space $\mathbb{P}_{\text{Berk}}^1$ is both compact and connected means in practice that many of the difficulties encountered in “classical” non-Archimedean dynamics disappear when one defines the Fatou and Julia sets as subsets of $\mathbb{P}_{\text{Berk}}^1$. For example, the notion of a connected component is straightforward in the Berkovich setting, so one avoids the subtle issues involved in defining Fatou components in $\mathbb{P}^1(\mathbb{C}_p)$ (e.g., the D -components versus analytic components in Benedetto’s paper [14], or the definition by Rivera-Letelier in [83]).

Suppose $\varphi \in K(T)$ is a rational function of degree $d \geq 2$. In §10.1, we construct a *canonical probability measure* μ_φ on $\mathbb{P}_{\text{Berk}}^1$ attached to φ , whose properties are analogous to the well-known measure on $\mathbb{P}^1(\mathbb{C})$ first defined by Lyubich and by Freire, Lopes, and Mañé. The measure μ_φ is φ -invariant (i.e., satisfies $\varphi_*(\mu_\varphi) = \mu_\varphi$) and also satisfies the functional equation $\varphi^*(\mu_\varphi) = d \cdot \mu_\varphi$.

In §10.2, we prove an explicit formula and functional equation for the Arakelov-Green’s function $g_{\mu_\varphi}(x, y)$ associated to μ_φ . These results, along with the Energy Minimization Principle mentioned earlier, play a key role in applications of the theory to arithmetic dynamics over global fields (see [5] and [9]). In §10.3, we use these results to prove an adelic equidistribution theorem (Theorem 10.24) for the Galois conjugates of algebraic points of small dynamical height over a number field k .

We then discuss analogues for $\mathbb{P}_{\text{Berk}}^1$ of classical results in the Fatou-Julia theory of iteration of rational maps on $\mathbb{P}^1(\mathbb{C})$. In particular, we define the Berkovich Fatou and Julia sets of φ and prove that the Berkovich Julia set J_φ (like its complex counterpart, but unlike its counterpart in $\mathbb{P}^1(K)$) is always nonempty. We give a new proof of the Favre–Rivera-Letelier equidistribution theorem for iterated pullbacks of Dirac measures attached to nonexceptional points, and using this theorem, we show that the Berkovich Julia set shares many properties with its classical complex counterpart. For example, it is either connected or has uncountably many connected components, repelling periodic points are dense in it, and the “Transitivity Theorem” holds.

In $\mathbb{P}^1(K)$, the notion of equicontinuity leads to a good definition of the Fatou set. In $\mathbb{P}_{\text{Berk}}^1$, as was pointed out to us by Rivera-Letelier, this remains true when $K = \mathbb{C}_p$ but fails for general K . We explain the subtleties regarding equicontinuity in the Berkovich case and give Rivera-Letelier’s proof that over \mathbb{C}_p the Berkovich equicontinuity locus coincides with the Berkovich Fatou set. We also give an overview (mostly without proof) of some of Rivera-Letelier’s fundamental results concerning rational dynamics over \mathbb{C}_p . While some of Rivera-Letelier’s results hold for arbitrary K , others make special use of the fact that the residue field of \mathbb{C}_p is a union of finite fields.

Appendices. In Appendix A, we review some facts from real analysis and point-set topology which are used throughout the text. Some of these (e.g., the Riesz Representation Theorem) are well known, while others (e.g., the Portmanteau theorem) are hard to find precise references for. We have provided self-contained proofs for the latter. We also include a detailed discussion of nets in topological spaces: since the space $\mathbb{P}_{\text{Berk}, K}^1$ is not in general metrizable, sequences do not suffice when discussing notions such as continuity.

In Appendix B, we discuss \mathbb{R} -trees and their relation to Gromov's theory of hyperbolic spaces. This appendix serves two main purposes. On the one hand, it provides references for some basic definitions and facts about \mathbb{R} -trees which are used in the text. On the other hand, it provides some intuition for the general theory of \mathbb{R} -trees by exploring the fundamental role played by the Gromov product, which is closely related to our generalized Hsia kernel.

Appendix C gives a brief overview of some basic definitions and results from Berkovich's theory of non-Archimedean analytic spaces. This material is included in order to give the reader some perspective on the relationship between the special cases dealt with in this book (the Berkovich unit disc, affine line, and projective line) and the general setting of Berkovich's theory.

Notation

We set the following notation, which will be used throughout unless otherwise specified. Symbols are listed roughly in the order they are introduced in the book, except that related notations are grouped together.

\mathbb{Z}	the ring of integers.
\mathbb{N}	the set of natural numbers, $\{n \in \mathbb{Z} : n \geq 0\}$.
\mathbb{Q}	the field of rational numbers.
$\overline{\mathbb{Q}}$	a fixed algebraic closure of \mathbb{Q} .
\mathbb{R}	the field of real numbers.
\mathbb{C}	the field of complex numbers.
\mathbb{Q}_p	the field of p -adic numbers.
\mathbb{Z}_p	the ring of integers of \mathbb{Q}_p .
\mathbb{C}_p	the completion of a fixed algebraic closure of \mathbb{Q}_p for some prime number p .
\mathbb{F}_p	the finite field with p elements.
$\overline{\mathbb{F}_p}$	a fixed algebraic closure of \mathbb{F}_p .
K	a complete, algebraically closed non-Archimedean field.
K^\times	the set of nonzero elements in K .
$ \cdot $	the non-Archimedean absolute value on K .
$\ x, y\ $	the spherical distance on $\mathbb{P}^1(K)$ associated to $ \cdot $, and also the spherical kernel, its canonical upper semicontinuous extension to $\mathbb{P}_{\text{Berk}}^1$ (see §4.3).
$\ (x, y)\ $	the norm $\max(x , y)$ of a point $(x, y) \in K^2$ (see §10.1).
q_v	a fixed real number greater than 1 associated to K , used to normalize $ \cdot $ and $\text{ord}_v(\cdot)$.
$\log_v(t)$	shorthand for $\log_{q_v}(t)$.
$\text{ord}_v(\cdot)$	the normalized valuation $-\log_v(\cdot)$ associated to $ \cdot $.
$ K^\times $	the value group of K , that is, $\{ \alpha : \alpha \in K^\times\}$.
\mathcal{O}	the valuation ring of K .
\mathfrak{m}	the maximal ideal of \mathcal{O} .
\tilde{K}	the residue field \mathcal{O}/\mathfrak{m} of K .
$\tilde{g}(T)$	the reduction, in $\tilde{K}(T)$, of a function $g(T) \in \mathcal{O}(T)$.
$K[T]$	the ring of polynomials with coefficients in K .
$K(T)$	the field of rational functions with coefficients in K .
$K[[T]]$	the ring of formal power series with coefficients in K .
$K\langle T \rangle$	the Tate algebra of formal power series converging on the closed unit disc.

\mathbb{A}^1	the affine line over K .
\mathbb{P}^1	the projective line over K .
$\mathbb{A}_{\text{Berk}}^1$	the Berkovich affine line over K .
$\mathbb{P}_{\text{Berk}}^1$	the Berkovich projective line over K .
\mathbb{H}_{Berk}	the “hyperbolic space” $\mathbb{P}_{\text{Berk}}^1 \setminus \mathbb{P}^1(K)$.
$\mathbb{H}_{\text{Berk}}^{\mathbb{Q}}$	the set of points of type II in \mathbb{H}_{Berk} (corresponding to rational discs in K).
$\mathbb{H}_{\text{Berk}}^{\mathbb{R}}$	the set of points of type II or III in \mathbb{H}_{Berk} (corresponding to either rational or irrational discs in K).
$\zeta_{a,r}$	the point of $\mathbb{A}_{\text{Berk}}^1$ corresponding to $D(a, r)$ under Berkovich’s classification theorem.
ζ_{Gauss}	the ‘Gauss point’ $\zeta_{0,1}$, corresponding to $D(0, 1)$.
$[\]_x$	the seminorm associated to a point $x \in \mathbb{P}_{\text{Berk}}^1$.
$[x, y]$	the path (or arc) from x to y .
$x \vee_{\zeta} y$	the point where the paths $[x, \zeta]$, $[y, \zeta]$ first meet.
$x \vee_{\infty} y$	the point where the paths $[x, \infty]$, $[y, \infty]$ first meet.
$x \vee y$	shorthand for $x \vee_{\zeta_{\text{Gauss}}} y$.
T_a	the ‘projectivized tangent space’ at $a \in \mathbb{P}_{\text{Berk}}^1$, the set of equivalence classes of paths emanating from a which share a common initial segment (see §B.6 in Appendix B).
$\vec{v} \in T_a$	a tangent direction at a (see §B.6 in Appendix B).
$\delta(x, y)_{\zeta}$	the generalized Hsia kernel with respect to ζ (see §4.4).
$\text{diam}_{\zeta}(x)$	the number $\delta(x, x)_{\zeta}$.
$\text{diam}_{\infty}(x)$	the number $\delta(x, x)_{\infty}$, equal to $\lim_{i \rightarrow \infty} r_i$ for any nested sequence of discs $\{D(a_i, r_i)\}$ corresponding to $x \in \mathbb{A}_{\text{Berk}}^1$.
$\text{diam}(x)$	the number $\ x, x\ = \delta(x, x)_{\zeta_{\text{Gauss}}}$.
$\rho(x, y)$	the path distance metric on \mathbb{H}_{Berk} ; see §2.7.
$\ell(Z)$	the total path length of a set $Z \subset \mathbb{H}_{\text{Berk}}$.
$j_{\zeta}(x, y)$	the fundamental potential kernel relative to the point ζ , given by $j_{\zeta}(x, y) = \rho(\zeta, x \vee_{\zeta} y)$.
\overline{X}	the closure of a set X in $\mathbb{P}_{\text{Berk}}^1$.
X^c	the complement $\mathbb{P}_{\text{Berk}}^1 \setminus X$.
∂X	the boundary of a set X .
$X(K)$	the set of K -rational points in X , i.e., $X \cap \mathbb{P}^1(K)$.
$\text{cl}_{\mathbb{H}}(X)$	the closure of a set $X \subset \mathbb{H}_{\text{Berk}}$, in the strong topology.
$\partial_{\mathbb{H}}(X)$	the boundary of a set $X \subset \mathbb{H}_{\text{Berk}}$, in the strong topology.
$D(a, r)$	the closed disc $\{x \in K : x - a \leq r\}$ of radius $r \geq 0$ centered at a . If $r \in K^{\times} $, we call the disc <i>rational</i> ; if $r \notin K^{\times} $, we call it <i>irrational</i> .
$D(a, r)^{-}$	the open disc $\{x \in K : x - a < r\}$ of radius r about a .
$\mathcal{D}(a, r)$	the closed Berkovich disc $\{x \in \mathbb{A}_{\text{Berk}} : [T - a]_x \leq r\}$ corresponding to the classical disc $D(a, r)$.
$\mathcal{D}(a, r)^{-}$	the open Berkovich disc $\{x \in \mathbb{A}_{\text{Berk}} : [T - a]_x < r\}$ corresponding to the classical disc $D(a, r)^{-}$.

$B(a, r)$	the closed ball $\{x \in \mathbb{P}^1(K) : \ x, a\ \leq r\}$ of radius r about a in $\mathbb{P}^1(K)$, relative to the spherical distance $\ x, y\ $.
$B(a, r)^-$	the open ball $\{x \in \mathbb{P}^1(K) : \ x, a\ < r\}$ of radius r about a in $\mathbb{P}^1(K)$, relative to the spherical distance.
$\mathcal{B}(a, r)_\zeta$	the closed ball $\{z \in \mathbb{P}_{\text{Berk}}^1 : \delta(x, y)_\zeta \leq r\}$.
$\mathcal{B}(a, r)_\zeta^-$	the open ball $\{z \in \mathbb{P}_{\text{Berk}}^1 : \delta(x, y)_\zeta < r\}$.
$\mathcal{B}(a, r)$	the closed ball $\{x \in \mathbb{P}_{\text{Berk}}^1 : \ x, a\ \leq r\} = \mathcal{B}(a, r)_{\zeta_{\text{Gauss}}}$.
$\mathcal{B}(a, r)^-$	the open ball $\{x \in \mathbb{P}_{\text{Berk}}^1 : \ x, a\ < r\} = \mathcal{B}(a, r)_{\zeta_{\text{Gauss}}}^-$.
$\mathcal{B}_a(\vec{v})^-$	the component of $\mathbb{P}_{\text{Berk}}^1 \setminus \{a\}$ corresponding to $\vec{v} \in T_a$.
$\widehat{\mathcal{B}}(a, \delta)$	the set $\{z \in \mathbb{H}_{\text{Berk}} : \rho(a, z) \leq \delta\}$, a closed ball for the strong topology.
$\widehat{\mathcal{B}}(a, \delta)^-$	the set $\{z \in \mathbb{H}_{\text{Berk}} : \rho(a, z) < \delta\}$, an open ball for the strong topology.
$\widehat{\mathcal{B}}_X(a, \delta)^-$	for $X \subset \mathbb{H}_{\text{Berk}}$, the set $X \cap \widehat{\mathcal{B}}(a, \delta)^-$.
$X(\zeta, \delta)$	the set $\{z \in \mathbb{P}_{\text{Berk}}^1 : \text{diam}_\zeta(x) \geq \delta\} = \widehat{\mathcal{B}}(\zeta, -\log_v(\delta))$.
Γ	a finite metrized graph.
$\text{CPA}(\Gamma)$	the space of continuous, piecewise affine functions on Γ .
$\text{Zh}(\Gamma)$	the Zhang space of Γ (see §3.4).
$\text{BDV}(\Gamma)$	the space of functions of ‘bounded differential variation’ on Γ (see §3.5).
$\langle f, g \rangle_{\Gamma, \text{Dir}}$	the Dirichlet pairing on Γ , for $f, g \in \text{BDV}(\Gamma)$.
$d_{\vec{v}}(f)$	the derivative of f in the tangent direction \vec{v} .
f'_+	the one-sided derivative of f in the positive direction along an oriented segment.
$f'_{\zeta, +}$	the derivative $d_{\vec{v}}(f)$ in the direction \vec{v} towards ζ .
$\Delta_\Gamma(f)$	the Laplacian of $f \in \text{BDV}(\Gamma)$.
$\Delta(f)$	in Chapter 3, the Laplacian $\Delta_\Gamma(f)$; elsewhere, $\Delta_{\mathbb{P}_{\text{Berk}}^1}(f)$.
$r_{U, X}$	the retraction map from a domain U to a closed subset X , often written r_X .
$(r_{U, X})_*(\mu)$	the pushforward from U to X of a measure μ , under $r_{U, X}$.
$\mathcal{C}(U)$	the space of continuous functions on U .
$\mathcal{C}(\overline{U})$	the space of continuous functions on the closure \overline{U} .
$\text{CPA}(U)$	the space of functions of the form $f \circ r_{U, \Gamma}$, with $f \in \text{CPA}(\Gamma)$ for some finite graph $\Gamma \subset U$.
$\text{BDV}(U)$	the space of functions of ‘bounded differential variation’ on a domain U (see §5.4).
$\mathcal{C}(\overline{U}) \cap \text{BDV}(U)$	by abuse of notation, the space of functions $f \in \mathcal{C}(\overline{U})$ with $f _U \in \text{BDV}(U)$ (see Definition 5.13).
$\mathcal{C}_c(U)$	the space of continuous functions vanishing outside a compact subset of U .
$\text{CPA}_c(U)$	the set of functions in $\text{CPA}(U)$ vanishing outside a compact subset of U .
$\text{BDV}_c(U)$	the set of functions in $\text{BDV}(U)$ vanishing outside a compact subset of U .

$\Delta_{\overline{U}}(f)$	the complete Laplacian of $f \in \text{BDV}(U)$ (see §5.4).
$\Delta_U(f)$	the Laplacian $\Delta_{\overline{U}}(f) _U$ of $f \in \text{BDV}(U)$ (see §5.4).
$\Delta_{\partial U}(f)$	the boundary derivative $\Delta_{\overline{U}}(f) _{\partial U}$ of $f \in \text{BDV}(U)$.
$\langle f, g \rangle_{U, \text{Dir}}$	the Dirichlet pairing on a domain U .
λ	the one-dimensional Hausdorff measure on \mathbb{H}_{Berk} , which restricts to dx on each segment.
$\text{supp}(\mu)$	the support of a measure μ .
$ \mu $	the measure $\mu_1 + \mu_2$, if the Jordan decomposition of the measure μ is $\mu_1 - \mu_2$.
$I_\zeta(\mu)$	the ‘energy integral’ $\iint -\log_v(\delta(x, y)_\zeta) d\mu(x)d\mu(y)$ for μ .
$u_\mu(z, \zeta)$	the potential function $\int -\log_v(\delta(x, y)_\zeta) d\mu(y)$ for μ .
$V_\zeta(E)$	the Robin constant of a set E , relative to the point ζ .
$\gamma_\zeta(E)$	the logarithmic capacity of a set E , relative to ζ .
μ_E	the equilibrium distribution of a set E .
$u_E(z, \zeta)$	the potential function associated to μ_E .
$G(x, \zeta; E)$	the Green’s function of a set E of positive capacity.
$d_\infty(E)_\zeta$	the transfinite diameter of a set E relative to ζ (see §6.4).
$\text{CH}(E)_\zeta$	the Chebyshev constant of E relative to ζ (see §6.4).
$\text{CH}^*(E)_\zeta$	the restricted Chebyshev constant of E relative to ζ .
$\text{CH}^a(E)_\zeta$	the algebraic Chebyshev constant of E relative to ζ .
\mathbb{E}	an ‘adelic set’ $\prod_v E_v$ for a number field k (see §7.8).
\mathfrak{X}	a finite, Galois-stable set of points in $\mathbb{P}^1(\overline{k})$.
\mathcal{P}^n	the set of n -dimensional probability vectors.
$\Gamma(\mathbb{E}, \mathfrak{X})$	the global Green’s matrix of \mathbb{E} relative to \mathfrak{X} .
$V(\mathbb{E}, \mathfrak{X})$	the global Robin constant of \mathbb{E} relative to \mathfrak{X} .
$\gamma(\mathbb{E}, \mathfrak{X})$	the global capacity of \mathbb{E} relative to \mathfrak{X} .
$\mathcal{H}(U)$	the space of harmonic functions on an open set U .
$\mathcal{SH}(U)$	the space of subharmonic functions on an open set U .
f^*	the upper semicontinuous regularization of a function f .
$\mathcal{M}^+(U)$	the space of positive, locally finite Borel measures on an open set U .
$\mathcal{M}_1^+(U)$	the space of Borel probability measures on U .
$g_\mu(x, y)$	the Arakelov-Green’s function associated to a probability measure μ .
$\mathcal{AG}[\zeta]$	the space of Arakelov-Green’s functions having a singularity at ζ .
$\varphi(T)$	a rational function in $K(T)$.
$\varphi^*(\mu)$	the pullback of a measure μ by the rational function φ .
$\varphi_*(\mu)$	the pushforward of a measure μ by φ .
$\varphi^{(n)}(T)$	the n -fold iterate $\varphi \circ \cdots \circ \varphi$.
$\deg(\varphi)$	the degree of the rational function $\varphi(T) \in K(T)$.
$m_\varphi(a)$	the multiplicity of $\varphi(T)$ at $a \in \mathbb{P}_{\text{Berk}}^1$.
$m_\varphi(a, \vec{v})$	the multiplicity of $\varphi(T)$ at a in the tangent direction \vec{v} .
$r_\varphi(a, \vec{v})$	the rate of repulsion of $\varphi(T)$ at a in the direction \vec{v} .

$N_\beta(V)$	the number of solutions to $\varphi(z) = \beta$ in V , counting multiplicities.
$N_{\zeta,\beta}^+(V)$	the number $\max(0, N_\zeta(V) - N_\beta(V))$.
$\mathcal{A}_{a,c}$	the open Berkovich annulus with boundary points a, c .
$\text{Mod}(\mathcal{A})$	the modulus of an annulus \mathcal{A} .
μ_φ	the ‘canonical measure’ associated to $\varphi(T)$.
$g_\varphi(x, y)$	another name for the Arakelov-Green’s function $g_{\mu_\varphi}(x, y)$ (see §10.2).
$\hat{h}_{\varphi,v,(x)}$	the Call-Silverman local height function associated to $\varphi(T)$ and the point x .
H_F	the homogeneous dynamical height function associated to $F = (F_1, F_2)$, where $F_1, F_2 \in K[X, Y]$ are homogenous polynomials.
$\text{Res}(F)$	the resultant of the homogeneous polynomials F_1, F_2 .
$\text{GO}(x)$	the ‘grand orbit’ of a point $x \in \mathbb{P}_{\text{Berk}}^1$ under $\varphi(T) \in K(T)$.
E_φ	the ‘exceptional set’ of all points in $\mathbb{P}_{\text{Berk}}^1$ having finite grand orbit under $\varphi(T)$.
F_φ	the Berkovich Fatou set of $\varphi(T)$.
J_φ	the Berkovich Julia set of $\varphi(T)$.
K_φ	the Berkovich filled Julia set of a polynomial $\varphi(T) \in K[T]$.
\mathbb{H}_p	the space \mathbb{H}_{Berk} , when $K = \mathbb{C}_p$.
$\mathcal{A}_x(\varphi)$	the immediate basin of attraction of an attracting fixed point x for $\varphi(T) \in \mathbb{C}_p(T)$.
$\mathcal{E}(\varphi)$	the domain of quasi-periodicity of a function $\varphi(T) \in \mathbb{C}_p(T)$.

Bibliography

- [1] E. Artin. *Algebraic numbers and algebraic functions*. AMS Chelsea Publishing, Providence, RI, 2006. Reprint of the 1967 original.
- [2] P. Autissier. Points entiers sur les surfaces arithmétiques. *J. Reine Angew. Math.*, 531:201–235, 2001.
- [3] M. Baker. A lower bound for average values of dynamical Green’s functions. *Math. Res. Lett.*, 13(2-3):245–257, 2006.
- [4] M. Baker. An introduction to Berkovich analytic spaces and non-Archimedean potential theory on curves. In *p-adic Geometry (Lectures from the 2007 Arizona Winter School)*, volume 45 of *AMS University Lecture Series*. Amer. Math. Soc., Providence, RI, 2008.
- [5] M. Baker. A finiteness theorem for canonical heights attached to rational maps over function fields. *J. Reine Angew. Math.*, 626:205–233, 2009.
- [6] M. Baker and L. DeMarco. Preperiodic points and unlikely intersections. Preprint. Available at [arXiv:0911.0918](https://arxiv.org/abs/0911.0918), 2009.
- [7] M. Baker and X. Faber. Metrized graphs, Laplacian operators, and electrical networks. In *Quantum graphs and their applications*, volume 415 of *Contemp. Math.*, pages 15–33. Amer. Math. Soc., Providence, RI, 2006.
- [8] M. Baker and L.-C. Hsia. Canonical heights, transfinite diameters, and polynomial dynamics. *J. Reine Angew. Math.*, 585:61–92, 2005.
- [9] M. Baker and R. Rumely. Equidistribution of small points, rational dynamics, and potential theory. *Ann. Inst. Fourier (Grenoble)*, 56(3):625–688, 2006.
- [10] M. Baker and R. Rumely. Harmonic analysis on metrized graphs. *Canad. J. Math.*, 59(2):225–275, 2007.
- [11] R. L. Benedetto. *Fatou components in p-adic dynamics*. PhD thesis, Brown University, Providence, RI, 1998.
- [12] R. L. Benedetto. *p*-adic dynamics and Sullivan’s no wandering domains theorem. *Compositio Math.*, 122(3):281–298, 2000.
- [13] R. L. Benedetto. Hyperbolic maps in *p*-adic dynamics. *Ergodic Theory Dynam. Systems*, 21(1):1–11, 2001.
- [14] R. L. Benedetto. Reduction, dynamics, and Julia sets of rational functions. *J. Number Theory*, 86(2):175–195, 2001.
- [15] R. L. Benedetto. Examples of wandering domains in *p*-adic polynomial dynamics. *C. R. Math. Acad. Sci. Paris*, 335(7):615–620, 2002.
- [16] V. G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., Providence, RI, 1990.
- [17] V. G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.*, 78:5–161, 1993.
- [18] V. G. Berkovich. The automorphism group of the Drinfel’d half-plane. *C. R. Acad. Sci. Paris Sér. I Math.*, 321(9):1127–1132, 1995.
- [19] V. G. Berkovich. *p*-adic analytic spaces. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, pages 141–151 (electronic), 1998.

- [20] V. G. Berkovich. Smooth p -adic analytic spaces are locally contractible. *Invent. Math.*, 137(1):1–84, 1999.
- [21] V. G. Berkovich. Smooth p -adic analytic spaces are locally contractible. II. In *Geometric aspects of Dwork theory*, pages 293–370. Walter de Gruyter and Co. KG, Berlin, 2004.
- [22] J.-P. Bézivin. Sur les points périodiques des applications rationnelles en dynamique ultramétrique. *Acta Arith.*, 100(1):63–74, 2001.
- [23] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.
- [24] Y. Bilu. Limit distribution of small points on algebraic tori. *Duke Math.*, 89:465–476, 1997.
- [25] E. Bombieri and W. Gubler. *Heights in Diophantine geometry*, volume 4 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2006.
- [26] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.
- [27] J.-B. Bost and A. Chambert-Loir. Analytic curves in algebraic varieties over number fields. Preprint. Available at [arXiv:math.NT/0702593](https://arxiv.org/abs/math.NT/0702593), 2008.
- [28] S. Boucksom, C. Favre, and M. Jonsson. Valuations and plurisubharmonic singularities. *Publ. Res. Inst. Math. Sci.*, 44(2):449–494, 2008.
- [29] N. Bourbaki. *General topology. Chapters 1–4*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French. Reprint of the 1989 English translation.
- [30] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.
- [31] H. Brolin. Invariant sets under iteration of rational functions. *Ark. Mat.*, 6:103–144, 1965.
- [32] G. S. Call and J. H. Silverman. Canonical heights on varieties with morphisms. *Compositio Math.*, 89(2):163–205, 1993.
- [33] D. G. Cantor. On an extension of the definition of transfinite diameter and some applications. *J. Reine Angew. Math.*, 316:160–207, 1980.
- [34] A. Chambert-Loir. Théorèmes d’équidistribution pour les systèmes dynamiques d’origine arithmétique. Panoramas et synthèses. Société Mathématique de France. Preprint.
- [35] A. Chambert-Loir. Mesures et équidistribution sur les espaces de Berkovich. *J. Reine Angew. Math.*, 595:215–235, 2006.
- [36] T. Chinburg, C. F. Lau, and R. Rumely. Capacity theory and arithmetic intersection theory. *Duke Math. J.*, 117(2):229–285, 2003.
- [37] T. Chinburg and R. Rumely. The capacity pairing. *J. Reine Angew. Math.*, 434:1–44, 1993.
- [38] I. Chiswell. *Introduction to Λ -trees*. World Scientific Publishing Co. Inc., River Edge, NJ, 2001.
- [39] B. Conrad. Several approaches to non-archimedean geometry. In *p -adic Geometry (Lectures from the 2007 Arizona Winter School)*, volume 45 of *AMS University Lecture Series*. Amer. Math. Soc., Providence, RI, 2008.
- [40] T. Coulbois, A. Hilion, and M. Lustig. Non-unique ergodicity, observers’ topology and the dual algebraic lamination for \mathbb{R} -trees. *Illinois J. Math.*, 51(3):897–911, 2007.
- [41] L. DeMarco. Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity. *Math. Ann.*, 326(1):43–73, 2003.

- [42] A. Ducros. Espaces analytiques p -adiques au sens de Berkovich. *Astérisque*, (311):Exp. No. 958, viii, 137–176, 2007. Séminaire Bourbaki. Vol. 2005/2006.
- [43] A. Escassut. *Analytic elements in p -adic analysis*. World Scientific Publishing Co. Inc., River Edge, NJ, 1995.
- [44] B. Farkas and B. Nagy. Transfinite diameter, Chebyshev constant and energy on locally compact spaces. *Potential Anal.*, 28(3):241–260, 2008.
- [45] C. Favre and M. Jonsson. *The valuative tree*, volume 1853 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.
- [46] C. Favre and J. Rivera-Letelier. Théorème d'équidistribution de Brolin en dynamique p -adique. *C. R. Math. Acad. Sci. Paris*, 339(4):271–276, 2004.
- [47] C. Favre and J. Rivera-Letelier. Équidistribution quantitative des points de petite hauteur sur la droite projective. *Math. Ann.*, 335(2):311–361, 2006.
- [48] C. Favre and J. Rivera-Letelier. Théorie ergodique des fractions rationnelles sur un corps ultramétrique. Preprint. Available at [arXiv:0709.0092](https://arxiv.org/abs/0709.0092), 2007.
- [49] G. B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1999. Modern techniques and their applications, a Wiley-Interscience Publication.
- [50] J. E. Fornæss and N. Sibony. Complex dynamics in higher dimension. I. *Astérisque*, (222):5, 201–231, 1994. Complex analytic methods in dynamical systems (Rio de Janeiro, 1992).
- [51] J. E. Fornæss and N. Sibony. Complex dynamics in higher dimensions. In *Complex potential theory (Montreal, PQ, 1993)*, volume 439 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 131–186. Kluwer Acad. Publ., Dordrecht, 1994. Notes partially written by Estela A. Gavosto.
- [52] J. E. Fornæss and N. Sibony. Complex dynamics in higher dimension. II. In *Modern methods in complex analysis (Princeton, NJ, 1992)*, volume 137 of *Ann. of Math. Stud.*, pages 135–182. Princeton Univ. Press, Princeton, NJ, 1995.
- [53] J. E. Fornæss and N. Sibony. Complex dynamics in higher dimension. In *Several complex variables (Berkeley, CA, 1995–1996)*, volume 37 of *Math. Sci. Res. Inst. Publ.*, pages 273–296. Cambridge Univ. Press, Cambridge, 1999.
- [54] A. Freire, A. Lopes, and R. Mañé. An invariant measure for rational maps. *Bol. Soc. Brasil. Mat.*, 14(1):45–62, 1983.
- [55] J. Fresnel and M. van der Put. *Rigid analytic geometry and its applications*, volume 218 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2004.
- [56] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.
- [57] É. Ghys and P. de la Harpe. Sur les groupes hyperboliques d'après Mikhael Gromov (Bern, 1988), volume 83 of *Progr. Math.*, Birkhäuser Boston, Boston, MA, 1990.
- [58] D. Goss. The algebraist's upper half-plane. *Bull. Amer. Math. Soc. (N.S.)*, 2(3):391–415, 1980.
- [59] E. Hille. *Analytic function theory. Vol. II*. Introductions to Higher Mathematics. Ginn and Co., Boston, Mass.-New York-Toronto, Ont., 1962.
- [60] N. R. Howes. *Modern analysis and topology*. Universitext. Springer-Verlag, New York, 1995.
- [61] L.-C. Hsia. Closure of periodic points over a non-Archimedean field. *J. London Math. Soc. (2)*, 62(3):685–700, 2000.
- [62] L.-C. Hsia. p -adic equidistribution theorems. Manuscript, 2003.
- [63] E. Kamke. *Theory of sets*. Dover, New York, 1950.
- [64] E. Kani. Potential theory on curves. In *Théorie des nombres (Quebec, PQ, 1987)*, pages 475–543. de Gruyter, Berlin, 1989.
- [65] J. L. Kelley. *General topology*. Springer-Verlag, New York, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.], Graduate Texts in Mathematics, No. 27.

- [66] J. Kiwi. Puiseux series polynomial dynamics and iteration of complex cubic polynomials. *Ann. Inst. Fourier (Grenoble)*, 56(5):1337–1404, 2006.
- [67] M. Klimek. *Pluripotential theory*, volume 6 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1991. Oxford Science Publications.
- [68] M. Kontsevich and Y. Soibelman. Affine structures and non-Archimedean analytic spaces. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 321–385. Birkhäuser Boston, Boston, MA, 2006.
- [69] H. Krause. Derived categories, resolutions, and Brown representability. In *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pages 101–139. Amer. Math. Soc., Providence, RI, 2007.
- [70] S. Lang. *Fundamentals of Diophantine geometry*. Springer-Verlag, New York, 1983.
- [71] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.
- [72] M. Lyubich. Entropy properties of rational endomorphisms of the Riemann sphere. *Ergodic Theory Dynam. Systems*, 3:351–385, 1983.
- [73] J. Milnor. *Dynamics in one complex variable*. Friedr. Vieweg & Sohn, Braunschweig, 1999. Introductory lectures.
- [74] P. Morton and J. H. Silverman. Periodic points, multiplicities, and dynamical units. *J. Reine Angew. Math.*, 461:81–122, 1995.
- [75] J. R. Munkres. *Topology: a first course*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1975.
- [76] J. Nicaise. Introduction to Berkovich spaces. Preprint. Available at wis.kuleuven.be/algebra/artikels/intro.berkovich.pdf.
- [77] K. Ohshika. *Discrete groups*, volume 207 of *Translations of Mathematical Monographs*. Amer. Math. Soc., Providence, RI, 2002. Translated from the 1998 Japanese original by the author, Iwanami Series in Modern Mathematics.
- [78] J. Piñeiro, L. Szpiro, and T. J. Tucker. Mahler measure for dynamical systems on \mathbf{P}^1 and intersection theory on a singular arithmetic surface. In Fedor Bogomolov and Yuri Tschinkel, editors, *Geometric Methods in Algebra and Number Theory*, volume 235 of *Progress in Mathematics*, pages 219–250. Birkhäuser, 2005.
- [79] T. Ransford. *Potential theory in the complex plane*, volume 28 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.
- [80] J. Rivera-Letelier. Théorie de Fatou et Julia dans la droite projective de Berkovich. In preparation.
- [81] J. Rivera-Letelier. Dynamique des fonctions rationnelles sur des corps locaux. *Astérisque*, (287):147–230, 2003. Geometric methods in dynamics. II.
- [82] J. Rivera-Letelier. Espace hyperbolique p -adique et dynamique des fonctions rationnelles. *Compositio Math.*, 138(2):199–231, 2003.
- [83] J. Rivera-Letelier. Sur la structure des ensembles de Fatou p -adiques. Preprint. Available at arXiv:math.DS/0412180, 35 pages, 2004.
- [84] J. Rivera-Letelier. Points périodiques des fonctions rationnelles dans l’espace hyperbolique p -adique. *Comment. Math. Helv.*, 80(3):593–629, 2005.
- [85] A. M. Robert. *A course in p -adic analysis*, volume 198 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.
- [86] H. L. Royden. *Real analysis*. Macmillan Publishing Company, New York, third edition, 1988.
- [87] W. Rudin. *Principles of mathematical analysis*. McGraw-Hill Book Co., New York, third edition, 1976. International Series in Pure and Applied Mathematics.
- [88] R. Rumely. *Capacity theory on algebraic curves*, volume 1378 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

- [89] R. Rumely. On Bilu's equidistribution theorem. In *Spectral problems in geometry and arithmetic (Iowa City, IA, 1997)*, volume 237 of *Contemp. Math.*, pages 159–166. Amer. Math. Soc., Providence, RI, 1999.
- [90] R. Rumely. The Fekete-Szegő theorem with splitting conditions. II. *Acta Arith.*, 103(4):347–410, 2002.
- [91] R. Rumely and M. Baker. Analysis and dynamics on the Berkovich projective line. Preprint. Available at [arXiv:math.NT/0407433](https://arxiv.org/abs/math.NT/0407433), 150 pages, 2004.
- [92] P. Schneider. Points of rigid analytic varieties. *J. Reine Angew. Math.*, 434:127–157, 1993.
- [93] J. H. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer, New York, 2007.
- [94] A. Thuillier. *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov*. PhD thesis, University of Rennes, 2005. Preprint. Available at <http://tel.ccsd.cnrs.fr/documents/archives0/00/01/09/90/index.html>.
- [95] P. Tortrat. Aspects potentialistes de l'itération des polynômes. In *Séminaire de Théorie du Potentiel, Paris, No. 8*, volume 1235 of *Lecture Notes in Math.*, pages 195–209. Springer, Berlin, 1987.
- [96] M. Tsuji. *Potential theory in modern function theory*. Chelsea Publishing Co., New York, 1975. Reprinting of the 1959 original.
- [97] J. W. Tukey. *Convergence and Uniformity in Topology*. Annals of Mathematics Studies, no. 2. Princeton University Press, Princeton, N. J., 1940.
- [98] B. L. van der Waerden. *Algebra. Vol. I*. Springer-Verlag, New York, 1991. Based in part on lectures by E. Artin and E. Noether. Translated from the seventh German edition by Fred Blum and John R. Schulenberger.
- [99] X. Yuan. Big line bundles over arithmetic varieties. *Invent. Math.*, 173(3):603–649, 2008.
- [100] S. Zhang. Admissible pairing on a curve. *Invent. Math.*, 112(1):171–193, 1993.

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The authors first give a detailed description of the topological structure of the Berkovich projective line and then introduce the Hsia kernel, the fundamental kernel for potential theory. Using the theory of metrized graphs, they define a Laplacian operator on the Berkovich line and construct theories of capacities, harmonic and subharmonic functions, and Green’s functions, all of which are strikingly similar to their classical complex counterparts. After developing a theory of multiplicities for rational functions, they give applications to non-Archimedean dynamics, including local and global equidistribution theorems, fixed point theorems, and Berkovich space analogues of many fundamental results from the classical Fatou-Julia theory of rational iteration. They illustrate the theory with concrete examples and exposit Rivera-Letelier’s results concerning rational dynamics over the field of p -adic complex numbers. They also establish Berkovich space versions of arithmetic results such as the Fekete-Szegő theorem and Bilu’s equidistribution theorem.



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