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Absolute CM-Periods

Hiroyuki Yoshida

American Mathematical Society

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PREFACE

The central theme of this book is an invariant attached to an ideal class of a totally real algebraic number field. Using this invariant, we can construct the Stark-Shintani units and the periods of abelian varieties with complex multiplication simultaneously. These periods were studied by Shimura and he crystallized them as the period symbol. The highlight of this book is the construction of the absolute CM-period symbol and the description of the remarkable properties of this symbol.

I planned to write this book in 1997 after getting a substantial part of the results in chapters I, II, III and V. My motivation was twofold: One was to present the results and conjectures in a systematic and precise form convenient for further research. Second was to organize the material to make it accessible to graduate students providing many illustrative examples. I benefitted from lecturing on the subject at several universities during these years, and the experience is incorporated into the presentation of this book. I would like to thank Professors Jacques Tilouine and Michael Harris who arranged my stay in Paris for two months in 1999, during which time I could draft chapters II and III. Finally I would like to thank Professor Goro Shimura for his extremely useful advice and encouragement.

Kyoto, June 2003. Hiroyuki Yoshida

NOTATION AND TERMINOLOGY

For a set S, $|S|$ denotes the cardinality of S. If S is a disjoint union of a family of sets T_λ, $\lambda \in \Lambda$, we express this fact by $S = \sqcup_{\lambda\in\Lambda} T_\lambda$. For $x \in \mathbf{R}$, $[x]$ denotes the largest integer which does not exceed x. By $\mathbf{R}_+$, we denote the set of all positive real numbers. For a, $b \in \mathbf{C}$, we write $a \sim b$ if $b \neq 0$ and a/b is an algebraic number. By $\mathfrak{H}$, we denote the complex upper half plane. For a subset X of $\mathbf{R}^n$, we put $\partial X = X \setminus X^\circ$, where X° denotes the set of interior points of X.

We denote by $\zeta(s)$ the Riemann zeta function. Euler's constant is denoted by γ. For a function $f(x)$ and a nonnegative integer p, $f^{(p)}(x)$ denotes the p-th derivative of $f(x)$ with the convention $f^{(0)}(x) = f(x)$. This notation will be used only in chapter I and appendix II in an obvious context. For a function $f(s)$ which is meromorphic in a neighborhood of $a \in \mathbf{C}$, $\mathrm{Res}_{s=a} f(s)$ denotes the residue of $f(s)$ at a.

For a finite group G, $\widehat{G}$ denotes the set of all equivalence classes of irreducible representations of G on vector spaces over $\mathbf{C}$. For a subgroup H of G and a representation ψ of H, the induced representation from ψ is denoted by $\mathrm{Ind}_H^G \psi$. For an associative ring R with unit, $R^\times$ denotes the group of all invertible elements of R. By $M(m, n, R)$, we denote the set of all $m \times n$-matrices with entries in R. We abbreviate $M(n, n, R)$ to $M(n, R)$ and set $GL(n, R) = M(n, R)^\times$. For two fields F_1 and F_2 contained in a field K, $F_1 \vee F_2$ denotes the composite field of F_1 and F_2.

We fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ in $\mathbf{C}$. By an algebraic number field, we understand an algebraic extension of $\mathbf{Q}$ of finite degree contained in $\overline{\mathbf{Q}}$. We denote by ρ the complex conjugation.

Let F be a global field, i.e., an algebraic number field or a function field of dimension 1 with a finite field as the field of constants. For a place v of F, F_v denotes the completion of F at v. For $a \in F_v$, $|a|_v$ denotes the absolute value of a, i.e., we have $d(ax) = |a|_v dx$ for a Haar measure dx on F_v. By F_A and $F_A^\times$, we denote the adele ring and the idele group of F respectively. By F_∞ (resp. $F_\infty^\times$), we denote the infinite part of F_A (resp. $F_A^\times$). By $(F_A)_f$ (resp. $(F_A^\times)_f$), we denote the finite part of F_A (resp. $F_A^\times$). For an algebraic group G defined over F, G_A denotes the adelization of G and G_∞ denotes the infinite part of G_A. For a place v of F, G_v denotes the group of F_v-rational points of G. By a Hecke character of $F_A^\times$, we mean a continuous homomorphism of $F_A^\times$ into $\mathbf{C}^\times$ which is trivial on $F^\times$. For $x \in F_A$ and a place v of F, x_v denotes the v-component of x. The finite part of x is denoted by x_f. For $x \in F_A^\times$, $|x|_A$ denotes the idele norm of x

Let F be an algebraic number field. The ring of integers and the group of units of F are denoted by $\mathfrak{O}_F$ and E_F respectively. We denote by E_F^+ the group of all totally positive units of F. The group of roots of unity contained in F is denoted by W_F. We put $w_F = |W_F|$. The regulator, the different, the absolute discriminant and the class number of F are denoted by R_F, $\mathfrak{d}_F$, D_F and h_F respectively. We

denote by $I(F)$ the ideal group of F. For an integral ideal $\mathfrak{f}$ of F, $I_{\mathfrak{f}}(F)$ denotes the ideal group of F modulo $\mathfrak{f}$, i.e., the group of all fractional ideals relatively prime to $\mathfrak{f}$. For $x \in F_A^\times$, $\mathrm{div}(x)$ denotes the fractional ideal $\prod_v \mathfrak{p}_v^{a_v}$, where v extends over all finite places of F, $\mathfrak{p}_v$ is the prime ideal corresponding to v and $a_v = \mathrm{ord}_v x_v$ with the normalized additive valuation ord_v at v. By $r_1(F)$ and $r_2(F)$, we denote the number of real places and of imaginary places of F respectively. We denote by J_F the set of all isomorphisms of F into $\mathbf{C}$ and by I_F the free abelian group generated by J_F. For $a \in F$, $a \gg 0$ means that a is totally positive. We denote the maximal abelian extension of F in $\overline{\mathbf{Q}}$ by F_{ab}. For $a \in F_A^\times$, $[a, F] \in \mathrm{Gal}(F_{ab}/F)$ denotes the image of a under the Artin map. For an abelian extension L of F of finite degree and a fractional ideal $\mathfrak{a}$ of F, which is relatively prime to the conductor of L, $(\frac{L/F}{\mathfrak{a}})$ denotes the Artin symbol.

Let K be an algebraic number field which is an extension of F of finite degree. By $\mathrm{Res}_{K/F}$, we denote the restriction homomorphism from I_K to I_F. By $\mathrm{Inf}_{K/F}$, we denote the homomorphism from I_F to I_K such that, for $\sigma \in J_F$, $\mathrm{Inf}_{K/F}(\sigma)$ is the sum of all elements of J_K whose restrictions to F coincide with σ. The norm map and the trace map from K to F are denoted by $N_{K/F}$ and by $\mathrm{Tr}_{K/F}$ respectively. The relative discriminant (resp. the relative different) of K over F is denoted by $D(K/F)$ (resp. $\mathfrak{d}_{K/F}$).

By a CM-field, we understand a totally imaginary quadratic extension of a totally real algebraic number field. For a CM-filed K, $\Phi \in I_K$ is called a CM-type of K if $\Phi = \sum_{i=1}^n \sigma_i \in I_K$ with $\sigma_i \in J_K$ and if $\Phi + \Phi\rho$ is the sum of all elements in J_K. We often identify Φ with the set of isomorphisms $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ or with the representation of K into $M(n, \mathbf{C})$ sending $a \in K$ to the diagonal matrix with a^{σ_1}, a^{σ_2}, $\ldots$, a^{σ_n} on the diagonal entries. A point $z = (z_1, z_2, \ldots, z_n) \in \mathfrak{H}^n$ is called a CM-point if for a CM-field K, a CM-type $\Phi = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ and for $a \in K$, $z_i = a^{\sigma_i}$, $1 \leqq i \leqq n$ holds.

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