KNOTS, MOLECULES, AND THE UNIVERSE: AN INTRODUCTION TO TOPOLOGY

ERICA FLAPAN

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Preface

There are a quite a few excellent topology textbooks written for undergraduate mathematics majors in their junior or senior year of college. Such books usually begin with an introduction to point set topology, followed by geometric topics such as homotopy, covering spaces, knot theory, the classification of surfaces, or some area of applied topology. While not all undergraduate topology texts follow this precise structure, almost all presuppose experience with proofs and exposure to a rigorous definition of continuity. For math departments that do not have a large number of majors or have an inordinate burden of service courses to teach, offering a topology course which presupposes such a background is almost impossible. As a result, many mathematics majors graduate without having the opportunity to take a topology class.

In addition to textbooks written for math majors, there are a number of noteworthy intuitive introductions to topology intended for a wider audience. Such books are great for curious students and for the general public, but can be difficult to teach out of because they rarely include definitions, statements of theorems, or a sufficient number of elementary exercises (without solutions in the back).

What seems to be absent from the available topology books is one that is easy to teach from, includes a wide range of topics, has no prerequisites, and makes no assumptions about the mathematical sophistication or motivation of the reader. Such a book could be the basis for a course that would attract students to the math major who might not be excited by the analytical and algebraic arguments that they typically see in their math courses during the first two years of college.

This book is intended to fill this gap. It is an elementary introduction to geometric topology and its applications to chemistry, molecular biology, and cosmology. It does not assume any particular mathematical or scientific background, sophistication, or even motivation to study abstract mathematics. It is meant to be fun and engaging while at the same time drawing students in to learn about fundamental topological and geometric ideas. Though the book can be read and enjoyed by non-mathematicians, college students, or even eager high school students, it is intended to be used as an undergraduate textbook.

With this in mind, all of the concepts are introduced with explicit definitions and the theorems are clearly stated so that it is easy for an instructor to develop lectures and for students to refer to when doing the homework. Some theorems are proved formally, others are proved intuitively, and for those results that aren't proved, students are told in which advanced course they might expect to see a proof. In addition, each chapter concludes with numerous exercises which are at a level appropriate for students in their first two years of college, and whose solutions are only available to instructors.

In contrast with most topology textbooks, the style of the book is informal and lively. Though all of the definitions and theorems are explicitly stated, they are given in an intuitive rather than a rigorous form. For example, rather than introducing the formal definition of an isotopy, two knots are defined to be equivalent if one can be deformed to the other. This style of presentation allows students to develop intuition about topology and geometry without getting bogged down in technical details. In addition, the topics in the book were chosen for their visual appeal, which is highlighted by the abundance of illustrations throughout. In order to make the text more fun and engaging, early on it introduces a cast of characters who reappear in multiple chapters. This includes the 1-dimensional character A. Dash, the 2-dimensional characters A. Square and B. Triangle, and the 3-dimensional character A. 3D-Girl, each of whom is learning about geometry and topology in order to understand his or her own universe. The characters are illustrated with amusing pictures that entice readers to think about the experience from the character's point of view.

The book is divided into three parts corresponding to the three areas referred to in the title. Part 1, consisting of Chapters 1–8, concerns two and three dimensional universes, though there is a brief foray into the fourth dimension in Chapter 2. The goal of Part 1 is to develop techniques that enable creatures in a given space to visualize possible shapes for their universe, and to use topological and geometric properties to distinguish one such space from another.

The second part of the text, consisting of Chapters 9–11, provides an introduction to knots and links. Building on the ideas developed in Part 1, the emphasis in Part 2 is on deformations of knots and links, and the use of invariants to distinguish inequivalent knots and links. Tricolorability is introduced in Chapter 9 as an invariant which is relatively simple to understand and apply but does not distinguish many knots. Then Chapter 10 surveys a collection of invariants which are not hard to understand but in some cases are hard to compute, and in the exercises we ask the reader to figure out what invariants should be used to distinguish a given pair of knots or links. Finally, Chapter 11 introduces the Kauffman bracket and Jones polynomial, and provides a step-by-step explanation of how to compute them, as well as some theorems that illustrate their power.

The third part of the text, consisting of Chapters 12–15, presents applications of topology and geometry to chemistry and molecular biology. In particular, Chapter 12 compares the concepts of mirror image symmetry from geometric, topological, and chemical viewpoints. It also introduces the idea of using embeddings of graphs in 3-dimensional space as models of non-rigid molecules. Then Chapter 13 presents techniques that can be used to show that some non-rigid molecules are topologically distinct from their mirror images. Chapter 14 explores the topology and geometry of DNA, developing material about rational tangles as a way to model DNA recombination. Finally, Chapter 15 describes topologically complex protein structures, giving examples of knotted and linked proteins, as well as proteins containing non-planar graphs. While Chapter 13 depends on Chapter 12, Chapters 14 and 15 are independent of one another and of Chapters 12 and 13. Thus an instructor can choose which of the applied topics they would like to cover.

The three parts of the book make it usable as a textbook for several different types of courses. For example, Part 1 of the book could be the basis for a seminar for first year college students on a topic such as "Dimensions and the Shape of the Universe." Since the book is written with students in mind, the instructor could give daily reading assignments and then lead the class in a discussion of the material followed by breaking the class into small groups to work through some of the exercises. Such a seminar would also lend itself well to open ended writing projects on diverse topics: Describe a sport that could take place in a 2-dimensional universe. Pick a well-known game (for example chess, checkers, Othello, Hex, or Go), and explain how the strategy for the game would change if it were played on a flat torus or a flat Klein bottle. Design a 4-dimensional version of such a game, and explain how the strategy compares to the usual 2- or 3-dimensional version. Imagine the form of a 4-dimensional person, and explain what we would see if the person passed through our space in different ways. If we had a spaceship that could travel in any direction as fast and far as we would like, what evidence should the spaceship seek to help us figure out the shape of our universe? These are a few of the topics.

The book could serve as the text for a topology course for math majors with or without prerequisites. The instructor could either cover all three parts of the text, or pick and choose topics from each of the three parts and supplement with point set topics from a more traditional topology text. Another option for a course for math majors would be for the instructor to give lectures to introduce each chapter, and then have students take turns giving lectures on each section within a chapter. The sections were intentionally made short to enable students to read and understand a single section at a time.

The text could also be used for an interdisciplinary course linking topology with molecular biology and chemistry. In this case, the course would focus on Part 2 and Part 3 (comprising Chapters 9–15). Part 2 would give students enough background in knot theory to understand the applications to molecular symmetry and the topology of DNA and proteins developed in Part 3. Note that it is not necessary for students to read Part 1 in order to understand Part 2, nor is it necessary for students to have a background in biology or chemistry in order to understand Part 3.

A geometry course for future teachers could also be based on this text. Such courses often consist of an axiomatic approach to Euclidean and non-Euclidean geometry. However, approaching geometry this way can be tedious and may actually kill any natural interest the student has in geometry. By contrast, a course for future teachers based on this text would focus on developing geometric thinking and visual intuition together with problem solving and mathematical reasoning. Such a course would cover Part 1, culminating in Chapter 8 which presents Euclid's five Axioms and alternatives to the Fifth Axiom that are valid on a sphere and a hyperbolic plane. The instructor for such a course would want to present the material slowly in order to emphasize visualization and logical deduction throughout. The nearly 450 illustrations in the text are designed to help students develop their visual intuition, and the many exercises at the end of each chapter would give students necessary practice writing mathematical arguments. Furthermore, the exercises are designed to encourage students to explain their reasoning in complete sentences and integrate illustrations into their explanations whenever possible. This course could culminate with a project such as designing a museum exhibit featuring the student's choice of the ten most important images from the text with an introduction to the exhibit and short explanatory descriptions associated with each image.

The students could also be asked to include some 3-dimensional models constructed themselves.

Finally, because the text is readable by students and is self-contained, the book would be well suited for an independent study or senior year project. Also, because of its lack of prerequisites and its division into three parts and the subdivision of each chapter into short sections, it would work well as the textbook for a four-week winter term or a summer course.

We hope that at least in some small way this book contributes to one or more of the following large goals: it motivates some undergraduates to major in math; it motivates some math majors to study topology; it motivates some math students to read their textbooks; it convinces some future K–12 teachers that geometry is fun; or it convinces some future scientists, administrators, or government officials that even the purest areas of mathematics can have important applications. At the very least, we hope that you and your students enjoy reading it.

Acknowledgments

As is apparent from the eighteen co-authors listed on the cover, this book has an unusual history. It began as lecture notes I developed for an intensive course I taught at the Summer Mathematics Program (SMP) at Carleton College for eleven summers between 2000 and 2014. The SMP was a very successful four-week, NSF-funded program for women students in their first and second year of college, with the goal of motivating, encouraging, and mentoring these students to get PhD's in mathematics. The students in the program had some prior exposure to proofs, but there was no specific prerequisite material. The aim of my course was to introduce them to the beauty and applications of topology and geometry, with the hope that the visual types of thinking they would experience might motivate those who were not as excited by algebraic and analytical arguments to continue in mathematics.

My course was originally inspired by three books. I used ideas from Jeffrey Weeks' wonderful book *The Shape of Space* to introduce the geometry and topology of 2- and 3-manifolds; I used ideas from Colin C. Adams' excellent book *The Knot Book* to expose students to knot theory; and I used ideas from my own book *When Topology Meets Chemistry* to explore applications of topology to molecular structures. However, over the fourteen years that I taught the course, my lectures took on a life of their own, veering away from these books. In particular, my approach to 2- and 3-manifolds became more formal than that of Weeks, though some of the arguments and organization in Chapters 1 through 8 are similar to his. For example, the proof in Chapter 4 that a surface in an orientable 3-manifold is 1-sided if and only if it is non-orientable is essentially the same as his, simply because I couldn't imagine a better proof. My approach to knot theory became focused on invariants in contrast with Adams' comprehensive introduction to knot theory, and my approach to applications of topology became less rigorous and more light-hearted than that of *When Topology Meets Chemistry*, and included the topology of proteins which was not in my earlier book.

While my lecture notes worked well for the intensive course that I taught at SMP, I did not originally foresee them becoming a book that could be used more widely. However, my ideas changed when I led the Undergraduate Faculty Program (UFP) at the Park City Mathematics Institute (PCMI) in 2011. As the leader of the UFP, I was charged with guiding a group of faculty to collaboratively produce materials that could be used at a wide range of institutions. I helped select the 18 UFP participants largely based on their interest in developing an undergraduate topology course with few or no prerequisites. These faculty members, who ranged from post-docs to full professors, came from a great variety of institutions. Some of their institutions were small while others were large, some were private and others public, and some had significant numbers of students headed for graduate school

in mathematics, while others had few math majors, almost none of whom would go on to graduate school.

During the UFP, I divided my lectures notes into twelve parts that would become twelve chapters of the manuscript. I then divided the eighteen participants into six groups of three and assigned each group two parts of my lecture notes to make into two chapters, one in the first half of the book and the other in the second half. Starting with their parts of the notes, the groups added topics, examples, exercises, figures, and more complete explanations. In addition, all eighteen UFP participants helped to write solutions to the exercises that were in the 2011 draft of the manuscript. After the program was over, some of the participants continued working on the manuscript in various capacities including editing chapters and adding more exercises and solutions. Ultimately, I revised, edited, rewrote, and added to the final manuscript so that it would all be in the same style and level, and would flow well.

Below, I list the eighteen UFP participants together with their individual contributions to the manuscript during and after the program.

- **Maia Averett** co-wrote Chapters 2 and 9, and edited Chapter 8.
- **Lance Bryant** co-wrote Chapters 3 and 10, and edited solutions for exercises in Chapter 3.
- **Shea Burns** co-wrote Chapters 3 and 10.
- **Jason Callahan** co-wrote Chapters 3 and 10, edited Chapters 6, 7, and 11, co-created the table of contents and index, clarified throughout the book which proofs would be included in the manuscript and which would be seen in more advanced courses, standardized numbering of results throughout, and contributed the book's title.
- **Jorge Calvo** co-wrote Chapters 2 and 9, edited Chapter 5, created some of the most difficult figures, and formatted and standardized word usage in Chapters 1–7.
- **Marion Moore Campisi** co-wrote Chapters 4 and 11, and added and edited figures throughout.
- **David Clark** co-wrote Chapters 2, 9, 10, and 14, revised and added exercises to Chapters 9, 10, and 11, edited solutions for exercises in Chapters 4 and 14, added and edited figures throughout, ensured all figures complied with AMS graphics guidelines, helped develop ideas for the graphic on the cover, and compiled this list of contributions.
- **Vesta Coufal** co-wrote Chapters 5 and 13, edited Chapters 1 and 10, added exercises to Chapters 6 and 7, checked lengths of sections, created common preambles for the chapters, co-created the table of contents and index, created a driver file, and worked out the technical details of formatting the final manuscript.
- **Elizabeth Denne** co-wrote Chapters 6, 7, and 12, edited Chapters 6, 7, 12, and 14, edited solutions for exercises in Chapters 6, 7, and 12, added exercises to Chapter 9, added references and URLs, and proofread the entire manuscript twice.
- **Berit Givens** co-wrote Chapters 5 and 13, edited Chapters 1 and 3, and edited solutions for exercises in Chapters 6, 7, and 13.
- **McKenzie Lamb** co-wrote Chapters 4 and 11, and edited solutions for exercises in Chapter 8.

- **Emille Davie Lawrence** co-wrote Chapters 1 and 8, edited the preface, introduction, and Chapters 5–15, added exercises to Chapter 3, and designed the graphic for the cover.
- **Lew Ludwig** co-wrote Chapters 6, 7, and 12, edited Chapter 9, formatted the manuscript as a book, and compiled a list of URLs used in the book.
- **Cornelia Van Cott** co-wrote Chapters 1 and 8, edited Chapters 10 and 11, and edited solutions for exercises in Chapters 9, 10, and 11.
- **Leonard Van Wyk** co-wrote Chapters 6, 7, and 12.
- **Robin Wilson** co-wrote Chapters 4, 11, and 14, and created figures for the solutions.
- **Helen Wong** co-wrote Chapters 5, 8, and 13, edited Chapters 1, 3, 7, 8, 14, and 15. Helen has also been my unofficial sounding board and helper for so many aspects of this book that I cannot list them all.
- **Andrea Young** co-wrote Chapters 1 and 8, edited Chapter 4, and edited solutions for exercises in Chapters 1 and 8.

In addition to the work of the UFP participants, several undergraduate and graduate students contributed to the development of the manuscript and solutions. I list them below together with their affiliation at the time when they worked on the project.

- **Bryan Brown** (undergraduate at Pomona College) proofread a draft of the entire manuscript and the solutions.
- **Dwayne Chambers** (PhD student at Claremont Graduate University) provided technical support during PCMI, TeXed solutions and created figures for the solutions during PCMI.
- **Gabriella Heller** (undergraduate at Pomona College) wrote the first version of Chapter 15 and created the protein drawing using the program Pymol.
- **Indra Elizabeth Kumar** (undergraduate at Scripps College) created and edited figures throughout the manuscript. In particular, the pictures of A. 3D-Girl were designed and created by her.
- **Becky Patrais** (PhD student at University of Minnesota) proofread and edited solutions to exercises.
- **Benjamin Russell** (undergraduate at Carleton College) wrote and edited some solutions to exercises in Chapters 8 and 15.

I also want to take this opportunity to thank Jeff Weeks and Colin Adams. Much of the manuscript was inspired by their outstanding books. In addition, I am grateful to Jeff Weeks for encouraging me to pursue this project. When I was afraid this manuscript would be too similar to his book *The Shape of Space*, he insisted that what is important is bringing this material to more students, not who was first to do so. I am also indebted to Colin Adams for his suggestions along the way, and for helpful comments on the figures throughout the manuscript. I hope this book is anywhere near as clear and inspiring as the many wonderful books that Colin has written.

Finally, I want to thank Grace Hibbard for generously sharing her mother Helen Wong with me for the last year of this project. Last, but certainly not least, I want to thank Francis Bonahon and Laure Flapan, whose love and support sustained me throughout this project and always.

Erica Flapan

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The style of the book is informal and lively. Though all of the definitions and theorems are explicitly stated, they are given in an intuitive rather than a rigorous form, with several hundreds of figures illustrating the exposition. This allows students to develop intuition about topology and geometry without getting bogged down in technical details.

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