Higher Order Fourier Analysis

Terence Tao

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To Garth Gaudry, who set me on the road;
To my family, for their constant support;
And to the readers of my blog, for their feedback and contributions.

Contents

Preface

Traditionally, Fourier analysis has been focused on the analysis of functions in terms of linear phase functions such as the sequence $n \mapsto e(\alpha n) := e^{2\pi i \alpha n}$. In recent years, though, applications have arisen—particularly in connection with problems involving linear patterns such as arithmetic progressions—in which it has been necessary to go beyond the linear phases, replacing them to higher order functions such as quadratic phases $n \mapsto e(\alpha n^2)$. This has given rise to the subject of *quadratic Fourier analysis* and, more generally, to *higher order Fourier analysis*.

The classical results of Weyl on the equidistribution of polynomials (and their generalisations to other orbits on homogeneous spaces) can be interpreted through this perspective as foundational results in this subject. However, the modern theory of higher order Fourier analysis is very recent indeed (and still incomplete to some extent), beginning with the breakthrough work of Gowers [**Go1998**], [**Go2001**] and also heavily influenced by parallel work in ergodic theory, in particular, the seminal work of Host and Kra [**HoKr2005**]. This area was also quickly seen to have much in common with areas of theoretical computer science related to polynomiality testing, and in joint work with Ben Green and Tamar Ziegler [**GrTa2010**], [**GrTa2008c**], [**GrTaZi2010b**], applications of this theory were given to asymptotics for various linear patterns in the prime numbers.

There are already several surveys or texts in the literature (e.g. [**Gr2007**], [**Kr2006**], [**Kr2007**], [**Ho2006**], [**Ta2007**], [**TaVu2006**]) that seek to cover some aspects of these developments. In this text (based on a topics graduate course I taught in the spring of 2010), I attempt to give a broad tour of this nascent field. This text is not intended to directly substitute for the core papers on the subject (many of which are quite technical and lengthy), but focuses instead on basic foundational and preparatory material, and on the simplest illustrative examples of key results, and should thus hopefully serve as a companion to the existing literature on the subject. In accordance with this complementary intention of this text, we also present certain approaches to the material that is not explicitly present in the literature, such as the abstract approach to Gowers-type norms (Section 2.2) or the ultrafilter approach to equidistribution (Section 1.1.3).

There is, however, one important omission in this text that should be pointed out. In order to keep the material here focused, self-contained, and of a reasonable length (in particular, of a length that can be mostly covered in a single graduate course), I have focused on the combinatorial aspects of higher order Fourier analysis, and only very briefly touched upon the equally significant ergodic theory side of the subject. In particular, the breakthrough work of Host and Kra [**HoKr2005**], establishing an ergodic-theoretic precursor to the inverse conjecture for the Gowers norms, is not discussed in detail here; nor is the very recent work of Szegedy [**Sz2009**], [**Sz2009b**], [**Sz2010**], [**Sz2010b**] and Camarena-Szegedy [**CaSz2010**] in which the Host-Kra machinery is adapted to the combinatorial setting. However, some of the foundational material for these papers, such as the ultralimit approach to equidistribution and structural decomposition, or the analysis of parallelopipeds on nilmanifolds, is covered in this text.

This text presumes a graduate-level familiarity with basic real analysis and measure theory, such as is covered in [**Ta2011**], [**Ta2010**], particularly with regard to the "soft" or "qualitative" side of the subject.

The core of the text is Chapter 1, which comprises the main lecture material. The material in Chapter 2 is optional to these lectures, except for the ultrafilter material in Section 2.1 which would be needed to some extent in order to facilitate the ultralimit analysis in Chapter 1. However, it is possible to omit the portions of the text involving ultrafilters and still be able to cover most of the material (though from a narrower set of perspectives).

Acknowledgments

I am greatly indebted to my students of the course on which this text was based, as well as many further commenters on my blog, including Sungjin Kim, William Meyerson, Joel Moreira, Thomas Sauvaget, Siming Tu, and Mads Sørensen. These comments, as well as the original lecture notes for this course, can be viewed online at

`terrytao.wordpress.com/category/teaching/254b-higher-order-fourier-analysis/`

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Traditional Fourier analysis, which has been remarkably effective in many contexts, uses linear phase functions to study functions. Some questions, such as problems involving arithmetic progressions, naturally lead to the use of quadratic or higher order phases. Higher order Fourier analysis is a subject that has become very active only recently. Gowers, in groundbreaking work, developed many of the basic concepts of this theory in order to give a new, quantitative proof of Szemerédi's theorem on arithmetic progressions. However, there are also precursors to this theory in Weyl's classical theory of equidistribution, as well as in Furstenberg's structural theory of dynamical systems.


Courtesy of Reed Hutchinson/UCLA

This book, which is the first monograph in this area, aims to cover all of these topics in a unified manner, as well as to survey some of the most recent developments, such as the application of the theory to count linear patterns in primes. The book serves as an introduction to the field, giving the beginning graduate student in the subject a high-level overview of the field. The text focuses on the simplest illustrative examples of key results, serving as a companion to the existing literature on the subject. There are numerous exercises with which to test one's knowledge.

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