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Bennett Chow
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Preface

About the book. Ricci flow is a geometric and analytic evolution equation which we believe is related to physical reality. One of the underlying principles of science is unity. In Ricci flow we see the unity of geometry and analysis. It is also expected to exhibit unity with low-dimensional topology. In this book we emphasize the more geometric and analytic aspects of Ricci flow rather than the topological aspects. We also attempt to convey some of the relations and formal similarities between Ricci flow and other geometric flows such as mean curvature flow. The interaction of techniques and ideas between Ricci flow and other geometric flows is a two-way street. So we hope the reader with a more general interest in geometric flows will benefit from the usefulness of applying ideas originating in Ricci flow to the study of other geometric flows. We have not aimed at completeness, even in the realm of the limited material that we cover. A more extensive coverage of the subject of Ricci flow is planned in the book by Dan Knopf and one of the authors [**163**] and its multi-authored sequel [**153**].

At places we follow the informal style of lecture notes and have attempted to cover some of the basic material in a relatively direct and efficient way. At the same time we take the opportunity to expose the reader to techniques, some of which lie outside of the subject of Ricci flow *per se*, which he or she may find useful in pursuing research in Ricci flow. So, metaphorically speaking, this book is a hybrid between rushing to work in the morning on a cold and blistery winter day and a casual stroll through the park on a warm and sunny midsummer afternoon. As much as possible, we have attempted to construct the book so that the chapters, and in some cases, individual sections, are relatively independent. In this way we hope that the book may be used as both an introduction and as a reference. Exercises appear at various places in the book and solutions to selected exercises are collected at the end of the book. We have endeavored to include some open problems which are aimed at conveying to the reader what the limits of our current knowledge are and to point to some interesting directions. We have also attempted to give the appropriate references so that the reader may further pursue the statements and proofs of the various results. To use real estate jargon, we hope that the references are reliable, but they are not guaranteed; in particular, sometimes the references given may not be the first place where a particular result is proven.

The purpose of this book is to give an introduction to the Ricci flow for graduate students and mathematicians interested in working in the subject. We have especially targeted the audience of readers who are novices to the Ricci flow. In order to make the book suitable for beginners, we have included in the first chapter basic material on Riemannian geometry. Many exercises and open problems can also be found scattered throughout this book. Some new progress of Perelman will appear in [**153**]. For example, a detailed coverage of entropy and the reduced distance will be included there. Moreover, comparisons are made between the Ricci flow and the linear heat equation, mean curvature flow, and other geometric evolution equations whenever possible. We have found that the analogies between the Ricci flow and other related heat-type equations are important and suggestive. Some new material on Hamilton's singularity analysis (unpublished recent work of Hamilton) has also been incorporated into this book; more generally, we have attempted to make the exposition of Hamilton's singularity analysis more complete. However, this book does not intend to cover all, or even most, classical aspects of the Ricci flow and the choice of topics reflects the authors' limited knowledge in the subject and the authors' preferences. The references at the end of the book and the comments at the end of the chapters are by no means intended to be exhaustive or complete. We only list the references we are aware of and which are most related to the material treated in the book. Due to the limited knowledge of the authors, it is inevitable that we have missed some important related works; we apologize for any omissions.

Prerequisites. We have tried to make this book as self-contained as possible, given the subject matter. The reader is assumed to have a basic knowledge of differentiable manifolds. Chapter 1 is an introduction to Riemannian geometry, and further basic topics in Riemannian geometry and geometric analysis are given in Appendix A. The reader familiar with Riemannian geometry may skip Chapter 1. Throughout the book there is a reliance on tensor calculus. Geometric (comparison geometry) methods also play an important role in this book. This is especially evident in the study of singularities of the Ricci flow. For these reasons we have included a number of exercises intended to develop the reader's proficiency in carrying out local coordinate calculations involving tensors and also applying comparison geometry techniques.

Course suggestions. A one semester course might consist of Chapter 1, Sections 2–5, Chapter 2, Sections 2–3 and 5–7, Chapter 3, Sections 1–6, Chapter 4, Sections 1–4, Chapter 5, Sections 1 and 4–6, with the rest of the sections in Chapters 1 through 5 and Appendix A optional. The second semester topics might include Chapter 6, Sections 1–3 and 5–7, Chapter 7, Sections 1–3, Chapter 8, Sections 1–4, Chapter 9, Sections 1–3 and 5–6, and Chapter 10, Sections 1–5, with the rest of the sections in Chapters 6 through 11 and Appendix B optional. The order of Chapters 6 and 7 is essentially interchangeable. See also the 'overall structure of the book' and 'suggested course outline' flowcharts at the end of the guide for the reader.

A word from our sponsor. Like its cousin, *The Ricci Flow: An Introduction* by Dan Knopf and one of the authors [**163**], which we will, for the moment, nickname "g_{ij}" after the metric, the present book, which we will nickname "Γ_{ijk}" after the connection, comprises an introduction to the subject of Ricci flow. As the notation suggests, Γ_{ijk} is derived from g_{ij}. In the following we compare and contrast the "metric" and its "connection".

1. Substance.

In g_{ij} we begin by emphasizing the special geometries and their role in the development of Ricci flow. Then we give detailed and comprehensive treatments of short time existence, maximum principles, Ricci flow on surfaces, and Hamilton's '3-manifolds with positive Ricci curvature theorem'. The remainder of g_{ij} is primarily devoted to proving some classical results on Type I singularities and background on such topics as the strong maximum principle, derivative estimates, the statement of the compactness theorem, and elementary singularity analysis including how to dilate about singularities. In the appendices tensor calculus and basic comparison geometry are presented.

In Γ_{ijk} we begin by giving a review of those aspects of Riemannian geometry which are needed for the study of the Ricci flow. We then discuss the same basic topics of short time existence, maximum principle, and 3-manifolds with positive Ricci curvature. We present the topics of Sobolev inequalities, isoperimetric estimates, Perelman's no local collapsing, Ricci solitons, higher-dimensional Ricci flow, and the Ricci flow on noncompact manifolds. After this we give a comprehensive introduction to singularity analysis, mixing classical results with the application of the no local collapsing theorem. Ancient solutions are presented in detail including the classical theory and Perelman's classification of 3-dimensional noncompact shrinking solitons. Finally we develop differential Harnack inequalities and the space-time approach to Ricci flow which leads to Perelman's reduced distance function. In the appendices geometric analysis related to Ricci flow and topics in geometric evolution equations are presented.

2. Style. We would like to imagine g_{ij} as an attempt to write in the style of *jazz* and Γ_{ijk} as an attempt in the style of *rock 'n' roll* (with apologies to rhythm and blues, gospel, country and western, hip hop, etc.). In g_{ij} we dive right into Ricci flow and then proceed at a metric pace, taking the time to appreciate the intricacies and nuances of the melody and structure of the mathematical music. In Γ_{ijk}, after starting from more basic material, as a connection to Ricci flow, the tempo is slightly more upbeat. The recital is defined on a longer page interval, and consequently more ground is covered, with the intention of leading up to the forefront of mathematical research. In g_{ij} calculations are carried out in detail in the main body of the book whereas in Γ_{ijk} the details appear either in the main body or at the end of the book in the solutions to the exercises. With the addition of basic core material on Riemannian geometry and a substantial number of solved exercises, Γ_{ijk} is accessible to graduate students and suitable for use in a semester or year-long graduate course.

Ricci flow and geometrization. The subject of Hamilton's Ricci flow lies in the more general field of geometric flows, which in turn lies in the even more general field of geometric analysis. Ricci flow deforms Riemannian metrics on manifolds by their Ricci tensor, an equation which turns out to exhibit many similarities with the heat equation. Other geometric flows, such as the mean curvature flow of submanifolds, demonstrate similar smoothing properties. The aim for many geometric flows is to produce canonical geometric structures by deforming rather general initial data to these structures. Depending on the initial data, the solutions to geometric flows may encounter singularities where at some time the solution can no longer be defined smoothly. For various reasons, in Ricci flow, the study of the qualitative aspects of solutions, especially ones which form singularities, is at present more amenable in dimension 3. This is precisely the dimension in which the Poincaré conjecture was originally stated; the higher-dimensional generalizations have been solved by Smale in dimensions at least 5 and by Freedman in dimension 4. Remaining in dimension 3, a vast generalization of this conjecture was proposed by Thurston, called the geometrization conjecture, which, roughly speaking, says that each closed

3-manifold admits a geometric decomposition, i.e., can be decomposed into pieces which admit complete locally homogeneous metrics.

Hamilton's program is to use Ricci flow to approach this conjecture. Perelman's work aims at completing this program. Through their works one hopes/expects that the Ricci flow may be used to *infer* the existence of a geometric decomposition by taking any initial Riemannian metric on any closed 3-manifold and proving enough analytic, geometric and topological results about the corresponding solution of the Ricci flow with surgery. Note on the other hand that, in this regard, one does *not* expect to need to prove the convergence of the solution to the Ricci flow with surgery to a (possibly disconnected) homogeneous Riemannian manifold. The reason for this is Cheeger and Gromov's theory of collapsing manifolds and its extension to the case where the curvature is only bounded from below. Furthermore, if one is only interested in approaching the Poincaré conjecture, then one does not expect to need the theory of collapsing manifolds. For these geometric and topological reasons, the study of the Ricci flow as an approach to the Poincaré and geometrization conjectures is reduced to proving certain analytic and geometric results. In many respects the Ricci flow appears to be a very natural equation and we feel that the study of its analytic and geometric properties is of interest in its own right. Independent of the resolution of the above conjectures, there remain a number of interesting open problems concerning the Ricci flow in dimension 3. In higher dimensions, the situation is perhaps even more interesting in that, in general, much less is known.

The year 1982 marked the beginning of Ricci flow with the appearance of Hamilton's paper on 3-manifolds with positive Ricci curvature. Since then, the development of Hamilton's program is primarily scattered throughout several of his papers (see [**81**] for a selection of Ricci flow papers edited by Cao, Chu, Yau and one of the authors). In Hamilton's papers (sometimes implicitly and sometimes by analogy) a well-developed theory of Ricci flow is created as an approach toward the geometrization conjecture. We encourage the reader to go back to these original papers which contain a wealth of information and ideas. Hamilton's program especially takes shape in the three papers [**287**], [**290**] and [**291**]. The first two papers discuss (among other important topics) singularity formation, the classification of singularities, applications of estimates and singularity analysis to the Ricci flow with surgery. The third paper discusses applications of the compactness theorem, minimal surface theory and Mostow rigidity to obtain geometric decompositions of 3-manifolds via Ricci flow under certain assumptions.

The recent spectacular developments due to Grisha Perelman aimed at completing Hamilton's program appear in [**452**] and [**453**]. Again the reader is encouraged to go directly to these sources which contain a plethora of ideas. Perelman's work centers on the further development of singularity and surgery theory. Of primary importance in this regard is Perelman's reduced distance function which has its precursors in the work of Li and Yau on gradient estimates for the heat equation and of Hamilton on matrix differential Harnack inequalities for the Ricci flow. A main theme in Perelman's work is the use of comparison geometry, including the understanding of space-time distance, geodesics, and volume, to obtain estimates. These estimates build upon in an ingenious way the earlier gradient estimates of Li and Yau and of Hamilton and the volume comparison theorem of Bishop and Gromov. In a sense, Perelman has further strengthened the bridge between the partial differential equation and comparison approaches to differential geometry in the setting of Ricci flow.

For finite extinction time of the Ricci flow with surgery on 3-manifolds, which is aimed at proving the Poincaré conjecture using neither the theory of collapse nor most of Hamilton's 'nonsingular' techniques, see [**454**] and Colding and Minicozzi [**176**]. The relevant results on collapsing manifolds with only lower curvature bounds were announced in Perelman [**452**] (with earlier related work in [**451**]) and appear in Shioya and Yamaguchi [**504**]. For expositions of Perelman's work, see Kleiner and Lott [**342**], Sesum, Tian, and Wang [**495**] and Morgan [**419**] (there is also a discussion of [**453**] in Ding [**193**]). We also encourage the reader to consult these excellent expository sources which clarify and fill in the details for much of Perelman's work.[1]

The authors
December 6, 2005

[1]We also understand that there are forthcoming works of Cao and Zhu [**86**], Morgan and Tian [**421**], and Topping [**536**] on Ricci flow devoted primarily to Perelman's work. We encourage the reader to consult these works.

Acknowledgments

This book originated as notes by the authors for lectures on Hamilton's Ricci flow given in China during the summer of 2004. These lecture notes were written during the authors' stays at Fudan, East China Normal, Beijing, and Zhejiang Universities. We soon decided to expand these notes into a book which covers many more topics, including some of Perelman's recent results. The work on this book was completed when the authors were at their respective institutions, the University of California at San Diego and the University of Oregon.

The first author gave lectures at the Beijing University Summer School and the Summer School at the Zhejiang University Center of Mathematical Sciences and one lecture each at East China Normal University, Fudan University, and the Nankai Institute. The second author gave lectures at the Summer School at the Zhejiang University Center of Mathematical Sciences. The third author gave lectures at Fudan University and a lecture at East China Normal University.

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The first author would especially like to thank Dan Knopf for collaboration over the years on the book "The Ricci Flow: An Introduction" [**163**], published recently by the American Mathematical Society. It has been a fun ride and we have enjoyed and benefited from this collaboration quite a bit. We have tried to minimize the overlap between these two books; unfortunately, to retain some degree of completeness, a certain amount of overlap has been unavoidable. Parts of this book are derived from notes written in the 1990's. This author would like to thank editors Ed Dunne, Sergei Gelfand, and Ina Lindemann at the AMS for their help with the AMS Ricci flow series of books and especially to Ed Dunne for all of his help in publishing his book "The Ricci flow: An Introduction" [**163**] with Dan Knopf and the future coauthored book "The Ricci flow: Techniques and Applications" [**153**]. This author would especially like to thank his teachers Professors Shing-Tung Yau and Richard Hamilton for their help, guidance and encouragement over many years. The benefit he has received from them encompasses much of his mathematical career.

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The second author dedicates this book to his teacher Professor Kung-Ching Chang on the occasion of his seventieth birthday.

The last author dedicates this book to both his wife, Jing-Jing Wu, and his teacher Professor Peter Li.

Bennett Chow
UC San Diego and East China Normal University

Peng Lu
University of Oregon

Lei Ni
UC San Diego

rcflow@math.ucsd.edu
December 6, 2005

A Detailed Guide for the Reader

Chapter 1. We present some basic results and facts from Riemannian geometry. The results in this chapter, for the most part, either are used or are analogous to results in the latter chapters on Ricci flow and other geometric flows. Although we have included in this chapter what we feel ideally the reader should know, he or she should be reassured that mastering all of the contents of this chapter is not a prerequisite for studying Ricci flow! Indeed, this chapter may be used as a reference to which the reader may refer when necessary.

In Section 2 we give a quick review of metrics, connections, curvature, and covariant differentiation. Of particular note are the Bianchi identities, the Lie derivative, and the covariant derivative commutator formulas in Section 3, which have applications to the formulas we shall derive for solutions of the Ricci flow. In Section 4 we recall the theory of differential forms and discuss the Laplace operator for tensors and Bochner formulas for differential forms. Since integration by parts is a useful technique in geometric analysis and in particular Ricci flow, in Section 5 we recall the divergence theorem and its consequences. We also give a quick review of the de Rham theorem and the Hodge decomposition theorem. In Section 6 we introduce the Weyl tensor and the decomposition of the Riemann curvature tensor into its irreducible components. We consider some basic aspects of locally conformally flat manifolds. In Section 7 we discuss Cartan's method of moving frames since it is a useful technique for computing curvatures, especially in the presence of symmetry. As an application, we give a proof of the Gauss-Bonnet formula for surfaces using moving frames. We also discuss hypersurfaces from the point of view of moving frames. Metric geometry has important implications in Ricci flow, so we discuss (Section 8) the first and second variation of arc length and energy of paths. Applications are Synge's theorem and the Hessian comparison theorem. We include the application of the second variation formula to long geodesics and a variational proof of the fact that Jacobi fields minimize the index form. We also discuss the first and second variation formulas for the areas of hypersurfaces. Such formulas have applications to minimal surface theory. In Section 9 we recall basic facts about the exponential map such as the Gauss lemma, the Hopf-Rinow theorem, Jacobi fields, conjugate and cut points, and injectivity radius estimates for positively curved manifolds. Next (Section 10) we present geodesic spherical coordinates using the exponential map. This is a convenient way of studying (Hessian) comparison theorems for Jacobi fields, and by taking the determinant, volume (Laplacian) comparison theorems. One may think of these calculations as associated to hypersurfaces (the distance spheres from a point) evolving in their normal directions with unit speed. (More generally, one may consider arbitrary speeds, including the mean curvature flow.) Observe that the Laplacian of the distance function is the radial derivative of the logarithm of the Jacobian, which is the mean curvature of the distance spheres. Similarly, the Hessian of the distance function is the radial derivative of the logarithm of the inner product of Jacobi fields, which is the second fundamental form of the distance spheres. We then begin to discuss in more detail the Laplacian and Hessian comparison theorems in Section 11. These results are essentially equivalent to the Bishop-Gromov volume and Rauch comparison theorems and have important analogues in Ricci flow. As an application, we prove the mean value inequality. In Section 12 we give detailed proofs of the Laplacian, volume and Hessian comparison theorems. In the case of the Laplacian comparison theorem, we prove the inequality holds in the sense of distributions. In Section 13 we discuss the Cheeger-Gromoll splitting theorem, which relies on the Busemann functions associated to a line being subharmonic (and hence harmonic), and the mean value inequality. We then discuss the Toponogov comparison theorem. Left-invariant metrics on Lie groups provide nice examples of solutions on Ricci flow which can often be analyzed, so we introduce some background material in Section 14. In the notes and commentary (Section 15) we review some basic facts about the first and second fundamental forms of hypersurfaces in Euclidean space, since we shall later discuss curvature flows of hypersurfaces to compare with Ricci flow.

Chapter 2. Here we begin the study of Ricci flow proper. Before we describe the contents of this chapter, we suggest that the reader may occasionally refer to Chapter 4 for some explicit examples of solutions to the Ricci flow; these examples may guide the reader's intuition when studying the abstract derivations throughout the book. We start in Section 1 with some historical remarks about geometric evolution equations. We also give a brief layman's description of how Ricci flow approaches the Thurston geometrization conjecture. The Ricci flow is like a heat equation for metrics. One quick way of seeing this (Section 2) is to compute the evolution equation for the scalar curvature using a variation formula we derive later. We get a heat equation with a nonnegative term. Because of this, we can apply the maximum principle (Section 3) to show that the minimum of the scalar curvature increases. The variation formula for the scalar curvature yields a short derivation of Einstein's equations as the Euler-Lagrange equation for the total scalar curvature (Section 4). By modifying the total scalar curvature, we are led to Perelman's energy functional. Next (Section 5) we carry out the actual computations of the variation of the connection and curvatures. When the variation is minus twice the Ricci tensor, we obtain the evolution equations for the connection, scalar and Ricci curvatures under the Ricci flow. This is the first place we encounter the Lichnerowicz Laplacian which arises in the variation formula for the Ricci tensor. Since the Ricci flow is a weakly parabolic equation, to prove short time existence, we use DeTurck's trick, which shows that it is equivalent to a strictly parabolic equation (Section 6). Having derived the evolution equations for the Ricci and scalar curvatures, we derive the evolution equation for the full Riemann curvature tensor (Section 7). This takes the form of a heat equation with a quadratic term on the right-hand side. In dimension 3, the form of the quadratic term is especially simple. In the notes and commentary (Section 8) we discuss the symbol of the linearization of the Ricci tensor.

Chapter 3. We give a proof of Hamilton's classification of closed 3-manifolds with positive Ricci curvature using Ricci flow. Hamilton's theorem (Section 1) says that under the normalized (volume-preserving) Ricci flow on a closed 3-manifold with positive Ricci curvature, the metric converges exponentially fast in every C^k-norm to a constant positive sectional curvature metric. The maximum principle for tensors (Section 2) enables us to estimate the Ricci and sectional curvatures. We first show that positive Ricci curvature and Ricci pinching are preserved. To control the curvatures, it is convenient to generalize the maximum principle for symmetric 2-tensors to a maximum principle for the curvature operator, and more generally, to systems of parabolic equations on a vector bundle (Section 3). Using this formalism, we show that the pinching of the curvatures improves and tends to constant curvature at points and times where the curvature tends to infinity. This is the central estimate in the study of 3-manifolds with positive Ricci curvature. Once we have a pointwise estimate for the curvatures, we need a gradient estimate for the curvature in order to compare curvatures at different points at the same time (Section 4). Based on the fact that the pinching estimate breaks the scale-invariance, we obtain a gradient of the scalar curvature estimate which shows that a scale-invariant measure tends to zero at points where the curvature tends to infinity. Combining the estimates of the previous sections, we show that indeed the curvatures tend to constant (Section 5). With control of the curvatures and assuming derivative estimates derived in a later chapter, we can show that the normalized Ricci flow converges exponentially fast in each C^k-norm to a constant positive sectional curvature metric (Section 6). In the notes and commentary (Section 7) we state some of the basic evolution equations under the mean curvature flow. These formulas are somewhat analogous to the equations for the Ricci flow.

Chapter 4. We discuss Ricci solitons, homogeneous solutions and other special solutions. From the study of singularity formation we are interested in solutions which exist for all negative time. Some of these solutions exist also for all positive time, which makes them even more special. Of fundamental importance are gradient Ricci solitons (Section 1). We carefully formulate the basic equations of a gradient Ricci soliton and show that gradient solitons can be put in a canonical form. It is interesting that Euclidean space is not only a steady soliton but also a shrinking and an expanding soliton (Section 2). We then briefly discuss the cylinder shrinking soliton. For the Ricci flow on low-dimensional manifolds, it is particularly important to consider complete Ricci solitons on surfaces with positive curvature. The cigar soliton (Section 3) is such a soliton on the Euclidean plane, rotationally symmetric, asymptotic to a cylinder at infinity and with curvature decaying exponentially fast. We exhibit the cigar soliton in various coordinate systems. An interesting rotationally symmetric ancient solution on the 2-sphere is the Rosenau solution (Section 4). As time tends to negative infinity, the Rosenau solution looks like a pair of cigars, one at each end. Indeed, backward (in time) limits at the endpoints yield the cigar soliton. Next we describe an explicit rotationally symmetric expanding soliton on the plane with positive curvature (Section 5). The curvature also decays exponentially fast as the distance to the origin tends to infinity. Moving up one dimension, we obtain the Bryant soliton (Section 6). This is a rotationally symmetric steady gradient Ricci soliton on Euclidean 3-space with positive sectional curvature. Here the curvature decays inverse linearly and the metric is essentially asymptotic to a paraboloid at infinity. This means that at infinity the metric, after rescaling, limits to a cylinder in the same sense that a parabola in the plane, after rescaling, limits to two parallel lines after dilating about points which tend to infinity. Some of the most interesting explicit examples are homogeneous solutions (Section 7). We consider SU(2) and Nil and analyze the ODE which arises from Ricci flow. Here, since the isometry group acts transitively on the manifold, the Ricci flow reduces to a system of ordinary differential equations. In dimension 3, these systems are largely well understood. We also consider the Ricci flow of bi-invariant metrics on a compact Lie group. Under the Ricci flow, isometries persist (Section 8). That is, the isometry group is nondecreasing. It is interesting to ask if the isometry group remains constant under the Ricci flow. Analogous to the Ricci flow on surfaces is the curvature shortening flow (CSF) of plane curves (Section 9). The analogue of the cigar soliton is the grim reaper for the CSF. The Rosenau solution also has an analogue for the CSF.

Chapter 5. In this chapter we discuss monotonicity formulas which yield isoperimetric and volume ratio estimates. In Section 1 we discuss isoperimetric inequalities and their relation with Sobolev inequalities. We also derive the logarithmic Sobolev inequality from the L^2 Sobolev inequality. In Section 2 we study the evolution of the length of paths and geodesics in preparation for studying the evolution of the isoperimetric ratio on surfaces. Another proof, due to Hamilton, of the convergence of the Ricci flow on a 2-sphere uses a monotonicity formula for the isoperimetric ratio (Section 3). In Section 4 we give a proof of Perelman's no local collapsing theorem. In Section 5 we present geometric applications of the no local collapsing theorem. Of particular note are the consequent local injectivity radius estimate and the fact that the cigar can be ruled out as a finite time singularity model. The local injectivity radius estimate enables one to take limits of dilations of finite time singular solutions on closed manifolds. To facilitate the exposition of this, we present some preliminaries on the compactness theorem for solutions of the Ricci flow. In Section 6 we give a shorter proof of the classification of closed 3-manifolds with positive Ricci curvature using Perelman's no local collapsing theorem and the compactness theorem. Here we only prove sequential convergence instead of exponential convergence. In Section 7 we discuss Hamilton's isoperimetric estimate for Type I singular solutions in dimension 3.

Chapter 6. In this chapter we collect the analytic results and techniques which are useful for singularity analysis. Estimates for all of the derivatives of the curvature (Section 1) in terms of bounds on the curvature enable one to show that the solution exists as long as the curvature remains bounded (the long time existence theorem). Thus if a solution forms a singularity in finite time (i.e., cannot be continued past a finite time), then the supremum of the curvature is infinity. In Section 2 we present the local derivative of curvature estimates of W.-X. Shi. These estimates are fundamental in the study of singularities of the Ricci flow. Another basic tool to study singular solutions is a Cheeger-Gromov-type compactness theorem for a sequence of solutions of Ricci flow (Section 3). We present both the global and local versions of this result. What is needed for this sequence is a curvature bound and an injectivity radius estimate (to prevent collapsing). The sequences we usually consider arise from dilating a singular solution about a sequence of points and times with the times approaching the singularity time. An interesting application of compactness is Sesum's result (Section 4) that a solution exists as long as its Ricci curvature is bounded. When studying the formation of singularities on a 3-manifold, the Hamilton-Ivey estimate is particularly useful (Section 5). Roughly speaking, it says that at large curvature points, the largest sectional curvature is positive and much larger than any negative sectional curvature in magnitude. It implies that limits of dilations (which we call singularity models) have nonnegative sectional curvature. Since nonnegative sectional curvature metrics are rather limited geometrically and (especially) topologically, this is a crucial first step in the surgery theory for singular 3-dimensional solutions. The Hamilton-Ivey estimate also tells us that ancient 2- and 3-dimensional solutions with bounded curvature have nonnegative sectional curvature. Nonnegative sectional curvature in dimension 3 is a special case of nonnegative curvature operator in all dimensions, a curvature condition which is preserved under the Ricci flow. Such solutions satisfy the strong maximum principle (Section 6), which says that either the solution has (strictly) positive curvature operator or the holonomy reduces and the image and kernel of the curvature operator are constant in time and invariant under parallel translation. This rigidity result is especially powerful in dimension 3 (Section 7), where it implies that a simply connected nonnegative sectional curvature solution of the Ricci flow either has positive sectional curvature, splits as the product of a surface solution with positive curvature (which is topologically a 2-sphere or the plane) and a line, or is the flat Euclidean space. In the notes and commentary (Section 8) we note that the derivative estimates may be improved if we assume bounds on some derivatives of the curvature.

Chapter 7. In Section 1 we present the spherical space form theorem of Huisken, Margerin, and Nishikawa which says that for an initial Riemannian manifold with sectional curvatures pointwise sufficiently close to that of a constant positive curvature space, the normalized Ricci flow exists for all time and converges to a constant positive curvature metric. In Section 2 we outline the proof of Hamilton's classification of 4-manifolds with positive curvature operator. In higher dimensions (Section 3), a solution with nonnegative curvature operator either has positive curvature operator, a Kähler manifold with positive curvature operator on $(1,1)$-forms, or is a locally symmetric space. Except in dimension 4, where it was solved by Hamilton, it is an open problem whether a closed Riemannian manifold with positive curvature operator converges under the Ricci flow to a metric with constant positive sectional curvature (spherical space form). A potentially useful result in this regard, due to Tachibana, says that an Einstein metric with positive curvature operator has constant sectional curvature and an

Einstein metric with nonnegative curvature operator is locally symmetric. The maximum principle discussed in Chapter 2 extends to complete noncompact manifolds. In Section 4 we present some general results about the maximum principle on noncompact manifolds. In Section 5 we survey some results on complete solutions of the Ricci flow on noncompact manifolds.

Chapter 8. Here we begin our study of singularities. To study singularities (Section 1), one takes dilations about sequences of points and times where the time tends to the singularity time. The limit solutions of such sequences, if they exist, are ancient solutions. It is useful to distinguish singular and ancient solutions according to the rate of blowup of the curvature. Type I singularities blow up in finite time at the rate of the standard shrinking sphere. Type II singularities form more slowly in the sense that in terms of the curvature scale, the time to blow up is longer than that of Type I. On the other hand, as a function of time to blow up, the curvature of a Type II singularity is larger than that of a Type I singularity. We also give lower bounds (gap estimates) for the supremum of the curvature as a function of time for singular and ancient solutions. Given a singularity type, we describe ways of picking sequences of points and times about which to dilate (Section 2). Suitable choices of such sequences lead to ancient solution limits, called singularity models. Given an ancient solution, we can also dilate again about a sequence of points and times. We also discuss the prototype for a Type II singularity: the degenerate neckpinch. An interesting open problem is to show that Type II singularities indeed exist, a result which is known for the mean curvature flow. Since complete noncompact ancient solutions are important in the study of singularities, we begin the study of the geometry at infinity of such solutions under various hypotheses (Section 3). Some useful geometric invariants of the geometry at infinity are the asymptotic scalar curvature ratio and the asymptotic volume ratio. To study the geometry at infinity, a useful technique is dimension reduction (Section 4). Here we assume the asymptotic scalar curvature ratio (ASCR) is infinite, that is, the lim sup of the scalar curvature times the square distance to an origin is equal to infinity. Roughly speaking, this says that the scalar curvature has slower than quadratic decay. Using a point-picking-type argument, we find good sequences of points tending to spatial infinity. When ASCR is infinite and $\mathrm{Rm} \geq 0$, there exists a limit which splits as the product of a one lower-dimensional solution with a line. In the notes and commentary we note that numerical studies of a degenerate neckpinch have been carried out by Garfinkle and Isenberg (Section 5).

Chapter 9. Considering ancient solutions on 2-dimensional surfaces (Section 1), we show using the Harnack inequality that ancient solutions with bounded curvature whose maximum is attained in space in time must be isometric to the cigar soliton. On the other hand, using Hamilton's entropy monotonicity, one can show that Type I ancient solutions are compact and in fact isometric to the round shrinking sphere. Complementarily, Type II ancient solutions (such as the Rosenau solution) must have a backward limit which is the cigar soliton. Our discussion includes as a corollary the classification of ancient κ-solutions on surfaces. An interesting result is that noncompact ancient solutions on surfaces with time-dependent bounds on the curvature can be extended to eternal solutions. We conjecture that eternal solutions (without assuming the supremum of the curvature is attained) are cigar solitons. Optimistically, one can hope that the only complete ancient surface solutions with bounded curvature are rotationally symmetric and, in fact, belong to one of three types: either the cigar, or a surface with constant curvature, or the Rosenau solution. In Section 2 we discuss aspects of ancient solutions relating to their type. A Type I ancient solution with positive curvature operator has infinite asymptotic scalar curvature ratio. For Type II ancient solutions with positive curvature operators there exist backward limits which are steady gradient Ricci solitons. We also state and prove Perelman's theorem that positively curved ancient solutions have vanishing asymptotic volume ratio and infinite asymptotic scalar curvature ratio. In Section 3 we give proofs of Hamilton's results that *steady* gradient Ricci solitons have infinite asymptotic scalar curvature ratio and complete noncompact *expanding* gradient Ricci solitons with positive Ricci curvature have positive asymptotic volume ratio. In Section 4 we discuss a local injectivity radius estimate for steady gradient Ricci solitons not assuming they are κ-noncollapsed. In Section 5 we discuss what is known about 3-dimensional singularities from the classical theory as well as Perelman's no local collapsing theorem. In dimension 3, by dimension reduction, there exists a limit which splits as the product of a surface solution with a line. By the no local collapsing theorem, the surface solution cannot be the cigar soliton. By a previous result of Hamilton, the surface must then be a round shrinking 2-sphere. In the case of a Type I ancient solution, we either have a shrinking spherical space form or there exists a backward limit which is a cylinder (2-sphere product with a line). For ancient κ-solutions the latter cannot exist. We conjecture that the κ-noncollapsed condition can be removed. In dimension 3 we also have another basic result of Perelman, the nonexistence of noncompact shrinking gradient solitons with positive sectional curvature (Section 6). In Section 7 we summarize the results in the chapter and pose some conjectures about long-existing solutions, especially in dimensions 2 and 3.

Chapter 10. We discuss various differential Harnack estimates. These are sharp pointwise derivative estimates which usually enable one to compare a solution at different points in space and time. In Ricci flow they have applications toward the classification of singularity models. We begin with the heat equation (Section 1) and present the seminal Li-Yau estimate for positive solutions. For manifolds with nonnegative Ricci curvature, the estimate is sharp in the sense that equality is obtained for the fundamental solution on Euclidean space. Since the Li-Yau estimate has a precursor in the work of Yau on harmonic functions, we discuss the Liouville theorem for complete manifolds with nonnegative Ricci curvature, which relies on a gradient estimate. The Li-Yau estimate has a substantial extension to solutions of the Ricci flow, due to Hamilton. To describe this, we begin with surfaces, since the form of the inequality and its proof are much simpler in this case. In Section 2 we prove a differential Harnack estimate for surfaces with positive curvature. Ricci solitons again motivate the specific quantities we consider. A perturbation of these arguments enables one to prove a Harnack estimate for surfaces with variable signed curvature. Interestingly, the above (trace) inequality may be generalized to a matrix inequality which, roughly speaking, gives a lower bound for the Hessian of the logarithm of the curvature. In this sense, the Harnack inequalities are somewhat analogous to the Laplacian and Hessian comparison theorems of Riemannian geometry discussed in Chapter 1. A fact related to Hamilton's entropy is that in the space of metrics on a surface with positive curvature, its gradient is the matrix Harnack quantity, which in dimension 2 is a symmetric 2-tensor. Next, in dimension 2, we consider a 1-parameter family of Harnack inequalities for the Ricci flow coupled to a linear-type heat equation (Section 3). In one instance we have the Li-Yau inequality, and in another instance we have the linear trace Harnack estimate, which generalizes Hamilton's trace estimate. An open problem is to generalize this to higher dimensions. In the Kähler case, this has been accomplished by one of the authors. With these preliminaries we move on to Hamilton's celebrated matrix Harnack estimate for solutions of the Ricci flow with nonnegative curvature operator (Section 4). The specific Harnack quadratic under consideration is motivated directly from the consideration of expanding gradient Ricci solitons. The trace inequality (obtained from the matrix inequality by summing over an orthonormal basis) is particularly simple to state and in all dimensions is surprisingly similar to the 2-dimensional inequality. We show that the matrix inequality in dimension 2 is the same as the previous estimate which was in the form of a symmetric 2-tensor being nonnegative. The proof of the matrix estimate depends on a calculation, which at first glance looks quite complicated (Section 5). However, using the formalism of considering tensors as vector-valued functions on the principal frame bundle (either $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{O}(n)$), we can simplify the computations. The matrix Harnack quadratic is a bilinear form on the Whitney sum of the bundle of 1-forms with the bundle of 2-forms. It satisfies a heat-type equation with a quadratic nonlinear term analogous to the heat-type equation satisfied by the Riemann curvature tensor. Taking bases of 1-forms and 2-forms, one can exhibit the quadratic nonlinearity as a sum of squares when the curvature operator is nonnegative. This is the main reason for why the matrix Harnack inequality may be proved by a maximum principle argument. The trace Harnack inequality has a generalization to a Harnack inequality for nonnegative definite symmetric 2-tensors satisfying the Lichnerowicz Laplacian heat equation coupled to the Ricci flow (Section 6). This Harnack estimate generalizes the trace Harnack estimate. Using this, we give a simplified proof of Hamilton's result that ancient solutions with nonnegative bounded curvature operator which attain the supremum of their scalar curvatures are steady gradient Ricci solitons. Further investigating the linearized Ricci flow, we give a pinching estimate for solutions to the linearized Ricci flow on closed 3-manifolds. An open problem is to find applications of this general estimate, perhaps in conjunction with the linear trace Harnack estimate or other new estimates. In the notes and commentary (Section 7) we briefly discuss the matrix Harnack estimate for the heat equation, the Harnack estimate for the mean curvature flow, and some tools for calculating evolution equations associated to Ricci flow.

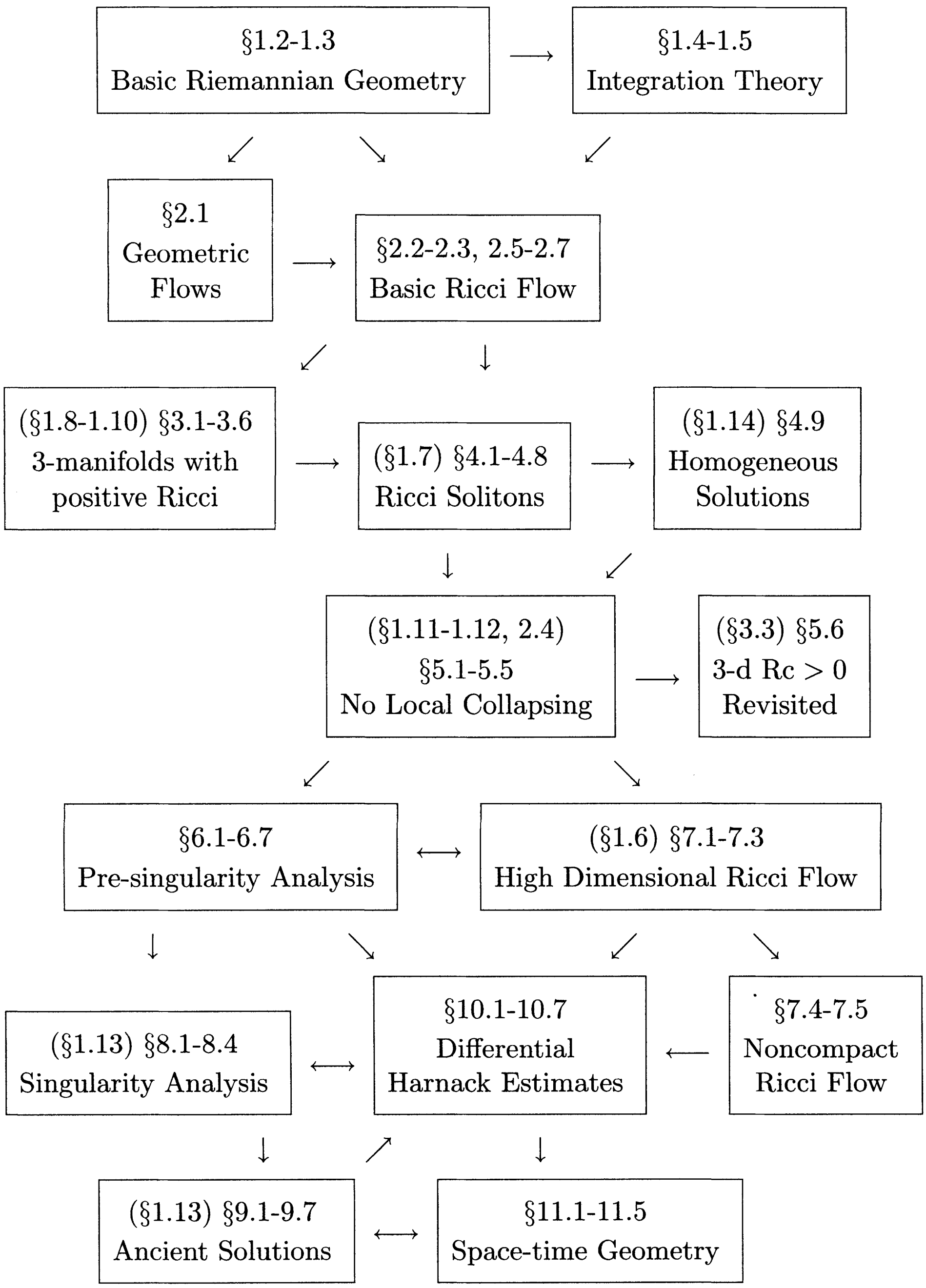
Chapter 11. We discuss various space-time geometries which are rather similar, culminating in Perelman's metric on the product of space-time with large-dimensional and large radius spheres. We begin with Hamilton's notion of the Ricci flow for degenerate metrics and the space-time connection of Chu and one of the authors (Section 1). This connection is compatible with the degenerate space-time metric and as a pair they satisfy the Ricci flow for degenerate metrics. We state the formulas for the Riemann and Ricci tensors of the space-time connection and observe curvature identities which suggest that the metric-connection pair is a Ricci soliton. We observe (Section 2) that the space-time Riemann curvature tensor is the matrix Harnack quadratic and the space-time Ricci tensor is the trace Harnack quadratic. Next we take the product of space-time with Einstein metric solutions of the Ricci flow (Section 3). We also introduce a scalar parameter into the definition of the potentially infinite space-time metric. We calculate the Levi-Civita connections of these metrics and observe that they essentially tend to the space-time connection defined in Section 1. It is particularly interesting that the space-time Laplacians tend to the heat (forward or backward) operator (depending on the sign of the scalar parameter). This fact depends on the dimension tending to infinity. Next we compute the Riemann and Ricci curvature tensors of the potentially infinite metrics. We observe that when the scalar parameter is -1, the metric tends to Ricci flat (this is due to Perelman). This is related to the developments on the ℓ-function discussed in [**153**]. We also recall the observation, again due to Perelman, that the metrics are essentially potentially Ricci solitons (at least he observed this when the parameter is either -1 or 1). Renormalizing the space-time length functional, we obtain the ℓ-function (Section 4). Not only is the space-time metric and connection related to the matrix Harnack estimate, it is also related to the linear trace Harnack estimate (Section 5). Here we need to make some modifications to describe the relation. An auxiliary function f is introduced and its definition is related to Perelman's idea of fixing the measure which he introduced when defining his energy and entropy. In the notes and commentary we discuss a space-time formulation of flows of hypersurfaces (Section 6) and its relation with Andrews' Harnack inequalities.

Appendix A. In this appendix we discuss various aspects of geometric analysis which are related to Ricci flow. In Section 1 we start with a short compendium of some inequalities used in the book. In Section 2 we recall some comparison theorems and explicit formulas for the heat kernel on Riemannian manifolds and in particular on constant curvature spaces. Next (Section 3) we discuss the Green's function, which is related to the heat equation and the geometry of the manifold. In Section 4 we give another proof of the Liouville Theorem 10.8. We also (Section 5) recall some basic facts about eigenvalues and eigenfunctions of the Laplacian. In the Ricci flow it is useful to study the lowest eigenvalue of certain elliptic (Laplace-like) operators. Li and Yau give a lower bound for the first eigenvalue on a closed manifold with nonnegative Ricci curvature. Lichnerowicz's result gives a lower bound assuming a positive lower bound of the Ricci curvature. We recall Reilly's formula and its application to estimating the first eigenvalue on manifolds with boundary. In Section 6 we discuss the definition of the determinant of the Laplacian via the zeta function regularization and compute the difference of determinants on a Riemann surface. The determinant of the Laplacian is an energy functional for the Ricci flow on surfaces. Since the Ricci flow evolves metrics and is related to the heat equation, we discuss (Section 7) the asymptotics of the heat kernel associated to the heat operator with respect to an evolving metric. The method is a slight modification of the fixed metric case. For comparison with Ricci flow monotonicity formulas, in Section 8 we recall some monotonicity formulas for harmonic functions and maps. In Section 9 we recall the Bieberbach theorem on the classification of flat manifolds; we approach the proof along the lines of Gromov's almost flat manifolds theorem.

Appendix B. In this appendix we discuss identities, inequalities, and estimates for various flows including the Ricci flow, Yamabe flow, and the cross curvature flow. It is interesting to compare these techniques. We begin in Section 1 by recalling the statement of the convergence result for the Ricci flow on closed surfaces. Next we consider the Kazdan-Warner and Bourguignon-Ezin identities in Section 2, from which it follows that a Ricci soliton on the 2-sphere has constant curvature. Then we turn our attention to Andrews' Poincaré-type inequality (Section 3), which holds in arbitrary dimensions. In dimension 2, it implies that Hamilton's entropy is monotone. This is useful in certain applications of the maximum principle, especially for systems. One of the proofs of the convergence of the Ricci flow on the 2-sphere relies on Ye's gradient estimate for the Yamabe flow of locally conformally flat manifolds with positive Yamabe invariant (Section 4). The Aleksandrov reflection method is used to obtain the gradient estimate. We also state Leon Simon's asymptotic uniqueness theorem. An interesting fact, which follows from the contracted second Bianchi identity, is that the identity map from a Riemannian manifold to itself, where the image manifold has the Ricci tensor as the metric (assuming it is positive), is harmonic (Section 5). In dimension 3 there is a symmetric curvature tensor, called the cross curvature tensor, which is dual to the Ricci tensor in the following sense. The identity map from a Riemannian 3-manifold to itself, where the domain manifold has the cross curvature tensor as the metric (assuming the sectional curvature is either everywhere negative or everywhere positive), is harmonic. We show two monotonicity formulas for the cross curvature flow which suggest that a negative sectional curvature metric on a closed 3-manifold should converge to a constant negative sectional curvature (hyperbolic) metric. In Section 6 we consider the time-derivative of the supremum function. This has applications to maximum principles. Finally, in the notes and commentary we note that in higher dimensions there are arbitrarily pinched negative sectional curvature metrics which do not support hyperbolic metrics.

Bibliography. We have included a number of references in the areas of Ricci flow, geometric evolution equations, geometric analysis, and related areas. Not all of the references are cited in the book. The references we do not cite are included so that the interested reader may be aware of various works in these fields which may be related to the topics discussed in this book.

Overall Structure of the Book



Suggested Course Outline

Semester 1

§1.2-1.5
Riemannian Geometry

↓

§2.2-2.3, 2.5-2.7
Basic Ricci Flow

↙

§3.1-3.6
3-manifolds with
positive Ricci

→

§4.1-4.4
Ricci Solitons

↓

§5.1, 5.4-5.6 No Local Collapsing

Semester 2

§6.1-6.3, 6.5-6.7
Pre-singularity Analysis

→

§7.1-7.3
High Dimensional Ricci Flow

↓

§8.1-8.4
Singularity Analysis

←

§9.1-9.3, 9.5-9.6
Ancient Solutions

↓

§10.1-10.5 Differential Harnack Estimates

Notation and Symbols

Here we list some of the notation and symbols used throughout the book.

Area	area of a surface or volume of a hypersurface
dA	area form (volume form in dimension 2)
ASCR	asymptotic scalar curvature ratio
AVR	asymptotic volume ratio
$B(p,r)$	ball of radius r centered at p
bounded curvature	bounded *sectional* curvature
const	constant
$\mathrm{Cut}(p)$	cut locus of p
Γ_{ij}^k	Christoffel symbols
∇	covariant derivative
$\doteqdot$	defined to be equal to
d	distance
div	divergence
$\cdot$	dot product or multiplication
int	interior
Δ, Δ_L, Δ_d	Laplacian, Lichnerowicz, Hodge Laplacians
L	length
LHS	left-hand side
H	mean curvature
$\mathrm{Hess}(f)$ or $\nabla\nabla f$	Hessian of f

$g(X,Y) = \langle X,Y \rangle$	metric or inner product
K_i	$\lvert \mathrm{Rm} \rvert (x_i, t_i)$
log	natural logarithm
$\mathcal{F}$, $\mathcal{W}$	Perelman's energy, entropy functional
$\mathcal{I}$	a time interval for the Ricci flow
$\mathcal{J}$	a time interval for the backward Ricci flow
ℓ	reduced distance or ℓ-function
$\mathcal{L}$	Lie derivative or $\mathcal{L}$-length
ν	unit outward normal
ODE / ODI	ordinary differential equation / inequality
PDE	partial differential equation
RF	Ricci flow
RHS	right-hand side
R, Rc, Rm	scalar, Ricci and Riemann curvature tensors
R_i	$R(x_i, t_i)$
h	second fundamental form
singularity model	a limit of dilations of a singular solution
$S(p,r)$	distance sphere
tr or Trace	trace
Vol	volume of a manifold
$d\mu$	volume form
$d\sigma$	volume form on boundary or hypersurface
ω_n	volume of the unit Euclidean n-ball
$n\omega_n$	volume of the unit Euclidean $(n-1)$-sphere
$W^{k,p}$	Sobolev space of functions with $\leq k$ derivatives in L^p
$W^{k,p}_{\mathrm{loc}}$	space of functions locally in $W^{k,p}$
$X^\flat$	dual 1-form to the vector field X
$\alpha^\natural$	dual vector field to the 1-form α

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