Algebraic Number Fields

Second Edition

Gerald J. Janusz

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To Gregory and Julie

—who have made their parents very proud.

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PREFACE TO THE SECOND EDITION

The main objective of this new edition remains the same as that of the first edition: to give an exposition of the introductory material and the main theorems about class fields of algebraic number fields while placing the minimum prerequisite demands on the reader. Although the basic outline remains the same, some new material has been added, many proofs have been expanded, and corrections have been made where errors in the first edition were found. Some of the additions to this edition are new treatments of some examples, emphasizing the use of the regular representation to do some calculations, a theorem determining the ring of integers in a number field with a finite amount of computation. The section on the topology of complete fields has been enlarged to correct a common error found in several texts regarding the existence of completions. A new section has been added giving the class number formula for quadratic fields. Some new exercises have been added. This edition was rekeyed (since no electronic version of the first edition exists) so that every line was rewritten.

The author expresses his thanks to the students and faculty at the University of Illinois, as well as many others, who have used this text and made comments regarding its improvement. Special thanks to Steve Ullom, Kurt Foster, Gennadii V. Matveev, and the late Irving Reiner, who sent comments and corrections to the first edition.

–GJJ
August 1995

Preface to the First Edition

This book contains an exposition of the main theorems of the class field theory of algebraic number fields along with the necessary introductory material. An attempt is made to keep the exposition self-contained. The only material presupposed is that which would be used in elementary Galois theory. The structure theorem for finitely generated modules over a principal ideal domain is used, but a proof is included in the appendix.

We use the direct approach to the subject by congruence subgroups of the ideal group rather than the more subtle description involving the cohomology of groups. This may be considered the historical approach to the subject, but we have presented it because it seems most useful for mathematicians who are specialists in other areas but wish to use it. From the student's point of view, this approach seems to require less background preparation and so is desirable for them.

The student who is not particularly interested in the theory of class fields can profitably read the first three chapters for an introduction to the study of arithmetic in fields, Dedekind domains, valuations, and general background material necessary for further work in several directions.

The first chapter contains an introduction to the algebra of number theory. The basic properties of Dedekind domains are presented using rather general ring theoretic arguments as much as possible. Emphasis is placed on local methods and proofs by localization. The results are given for rather general fields except in the last three sections. There we discuss cyclotomic extensions of the rational field and prove the unit theorem and the finiteness of the class number for algebraic number fields.

Valuations and complete fields are discussed in the second chapter. This depends partly on the previous chapter.

Chapter three contains material connecting Galois groups and ramification. The last section gives properties of the Artin map, which are of fundamental importance for the rest of the book.

In Chapter four the material becomes more specialized. We work exclusively with algebraic number fields and their completions. The analytic theorems proved include the Frobenius density theorem and Dirichlet's theorem on primes in arithmetic progression.

Chapter five contains the main results on class fields. A small amount of cohomology is developed here. We use only H^0 and H^1 and then only for cyclic groups. In this case a concrete description of the cohomology groups is used so the reader is not carried far from the fields and Galois groups. The approach to the main theorems makes systematic use of the Artin map and the Artin reciprocity theorem. We close the chapter with a discussion of the Hilbert class field and Artin's reduction of the principal ideal theorem.

The last chapter is intended primarily for illustration of the concepts introduced in the earlier chapters. We study mainly quadratic fields and prove a result which goes back to Gauss giving information about the class group of a quadratic number field. A few calculations are made to illustrate the use of the norm residue symbols.

The material in this text was used in a year-long course at the University of Illinois in 1970-1971. The first three chapters were covered in the first semester and the balance in the second semester.

REFERENCES

1. E. Artin, *Theory of Algebraic Numbers*, Lecture Notes, Göttingen, 1959.
2. E. Artin and J. Tate, *Class Field Theory*, Benjamin, New York, 1967.
3. A. Baker, *Linear forms in the logarithms of algebraic numbers*, Mathematika **13** (1966), 204–216.
4. Z. I. Borevich and I. R. Shafarevich, *Number Theory*, Academic Press, New York, 1967.
5. J. Cassels & A. Fröhlich, Eds., *Algebraic Number Theory*, Thompson Publ., Washington, D.C., 1967.
6. Ph. Furtwangler, *Beweis des Hauptidealsatzes für Klassenkörper algebraischer Zahlenkörper*, Abhand. Math. Seminar Hamburg **7** (1930), 14–36.
7. L. J. Goldstein, *Analytic Number Theory*, Prentice-Hall, Englewood Cliffs, N. J., 1971.
8. H. Hasse, *Vorlesungen über Klassenkörpertheorie*, Physica, Wurzburg, 1967.
9. E. S. Golod and I. R. Shafarevich, *On the class-field tower*, Izvestiya Akad. Nauk. SSSR **28** (1964), 261–272. (Russian)
10. Neal Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, Springer-Verlag, Berlin, Heidelberg, and New York, 1977.
11. Serge Lang, *Algebra*, Addison-Wesley, Reading, Mass., 1965.
12. Serge Lang, *Algebraic Number Theory*, Addison-Wesley, Reading, Mass., 1970.
13. James Munkres, *Topology*, Prentice Hall, Englewood Cliffs, NJ, 1975.
14. J. Neukirch, *Klassenkörpertheorie*, Universität Bonn, Bonn, 1967.
15. J. P. Serre, *Corps Locaux*, Hermann, Paris, 1962.
16. Andreas Speiser, *Die Zerlegungsgruppe*, J. für die Reine und Angew. Math. **149** (1919), 174–188.
17. H. M. Stark, *A complete determination of complex quadratic fields of class number one*, Mich. Math J. **14** (1967), 1–27.
18. A. Weil, *Basic Number Theory*, Springer-Verlag, Berlin, Heidelberg, and New York, 1967.
19. E. Weiss, *Algebraic Number Theory*, McGraw Hill, New York, 1963.

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