FIELDS INSTITUTE MONOGRAPHS

THE FIELDS INSTITUTE FOR RESEARCH IN MATHEMATICAL SCIENCES

Galois Module Structure

Victor P. Snaith

American Mathematical Society

http://dx.doi.org/10.1090/fim/002

Fields Institute Monographs

The Fields Institute for Research in Mathematical Sciences

Galois Module Structure

Victor P. Snaith

American Mathematical Society
Providence, Rhode Island

The Fields Institute for Research in Mathematical Sciences

The Fields Institute is named in honour of the Canadian mathematician John Charles Fields (1863–1932). Fields was a remarkable man who received many honours for his scientific work, including election to the Royal Society of Canada in 1909 and to the Royal Society of London in 1913. Among other accomplishments in the service of the international mathematics community, Fields was responsible for establishing the world's most prestigious prize for mathematics research—the Fields Medal.

The Fields Institute for Research in Mathematical Sciences is supported by grants from the Ontario Ministry of Education and Training and the Natural Sciences and Engineering Research Council of Canada. The Institute is sponsored by McMaster University, the University of Toronto, and the University of Waterloo and has affiliated universities in Ontario and across Canada.

The author received support for this volume from the Ontario Ministry of Education and Training and the Natural Sciences and Engineering Research Council of Canada.

Key words and phrases. Artin L-Function, Artin root number, Birch-Tate conjecture, Chinburg invariant, class group, Galois cohomology, Galois module structure, maximal order, p-adic L-function, representation theory.

1991 *Mathematics Subject Classification.* Primary 11R33;
Secondary 11M06, 11R23, 11R34, 11R65, 11R70, 11S25, 11S31, 11S40, 12G05, 16G99, 16H05, 19A31, 19D99, 19F27, 20J06.

Library of Congress Cataloging-in-Publication Data

Snaith, V. P. (Victor Percy), 1944–
Galois module structure/Victor P. Snaith.
p. cm. —(Fields Institute monographs, ISSN 1069-5273; 2)
At head of title: The Fields Institute for Research in Mathematical Sciences.
Includes bibliographical references and index.
ISBN 0-8218-0264-X
1. Galois modules (Algebra) 2. L-function. I. Fields Institute for Research in Mathematical Sciences. II. Title. III. Series.
QA247.S497 1994
512′.74–dc20 94-31144
CIP

This publication was prepared by the Fields Institute.

10 9 8 7 6 5 4 3 2 1 99 98 97 96 95 94

Contents

Preface

'The philosophers hitherto have only interpreted the world in various ways: the thing is, however, to change it.'
Karl Marx (1848)

This monograph is based upon a graduate course given at The Fields Institute for Research in Mathematical Sciences during the Autumn Term 1993. The objective of the course was to introduce an audience of advanced graduate students, Post-Doctoral Fellows and others familiar with basic algebraic number theory to some of the topics studied under the collective rubric of Galois Module Structure. As part of The Fields Institute's programme on L-functions the scope of the course was intended to familiarise the audience with the material sufficiently enough for them to be able to participate in the Galois Module Structure workshop which was held at The Fields Institute during February 14–18,1994. Accordingly I centred the material around one of the Chinburg conjectures because these conjectures are currently still open but are supported by a body of partial results interesting enough to illustrate a number of the subject's techniques in action.

The course was one of four given in connection with the L-functions programme and during the lectures I was able to benefit from and refer to the parallel lecture series for background. However, with future generations of graduate students in mind, I have tried to make the book self-contained in such a way as to be suitable for use as the basis for a graduate course. For this reason I have not necessarily battled through all the technicalities of the main results but instead have given proofs in special cases designed to illuminate rather than intimidate. For example, I have given a new proof of M.J. Taylor's theorem—the solution of the Fröhlich conjecture—but have restricted myself to the case of a group of odd prime power order. Similarly, in the case of D. Holland's theorem—the solution of the second Chinburg conjecture modulo $D(\mathbf{Z}[G])$—I have confined myself to the locally cyclic case with a brief indication of how to promote this special case to give the general proof.

With subsequent graduate courses in mind I have tried to include some 'interesting' exercises at the end of each chapter.

Now let me describe the contents in more detail.

Chapter One consists of basic material. Section 1.1 introduces the cohomology of finite groups, Tate cohomology and Galois cohomology. Section 1.2 introduces local fields and gives a quick, simple cohomological proof, which none of my audience seemed to have encountered previously, of the fundamental reciprocity isomorphism (Theorem 1.2.2) of local class field theory. This proof is based upon an exercise to be found in Serre [1979], p.202. Section 1.3 contains the elementary background on the complex representation theory of finite groups. The main new techniques which permeate the later chapters are all derived from the Explicit Brauer Induction homomorphism, a_G, which is described in this section. This is a natural, canonical form for Brauer's Induction Theorem, which originated in Snaith [1988] and Snaith [1989a] and was developed, modified and improved in Boltje [1990], Boltje [1989], Boltje, Snaith and Symonds [1992], Snaith [1989b], Snaith [1994] and Symonds [1991]. The important properties of a_G are collected, without proof, in Section 1.3. In Section 1.4 we recall the analytic prerequisites, namely Artin L-functions, Artin conductors, Artin root numbers, local Gauss sums, local root numbers and p-adic L-functions. In preparation for the intervention of these analytic invariants in the Hom-description class-groups, their behaviour under the Galois action of $\Omega_{\mathbf{Q}}$ is described. These calculations are facilitated by the use of the canonical form, a_G, for Brauer's Induction Theorem.

Chapter Two deals with the class-group, $\mathcal{CL}(\mathcal{O}[G])$, of an integral group-ring. In Section 2.1 we recall the Hom-description of the class-group of a group-ring. This describes the class-group in terms of a quotient of the idèlic-valued functions on the complex representation ring of G, $R(G)$. In Section 2.1 the Hom-description of the class-group of a maximal order of $\mathbf{Q}[G]$ is also described, in readiness for use in Chapter Five. The kernel of the homomorphism from $\mathcal{CL}(\mathbf{Z}[G])$ to the class-group of a maximal order is denoted by $D(\mathbf{Z}[G])$, whose Hom-description is also given in Section 2.1. These Hom-descriptions allow us to interpret analytic invariants defined on $R(G)$ as elements of $\mathcal{CL}(\mathbf{Z}[G])$ in the case when G is a Galois group. For example, if L/K is a Galois extension of number fields then the Artin root numbers of symplectic representations of the Galois group, $G(L/K)$, give rise to an analytic class, $W_{L/K} \in \mathcal{CL}(\mathbf{Z}[G(L/K)])$, which is defined in Section 2.2.1. On the other hand, in Section 2.2.3, using the fundamental classes of local class field theory of (1.2.3), we define the second Chinburg invariant, $\Omega(L/K, 2) \in \mathcal{CL}(\mathbf{Z}[G(L/K)])$. When L/K is tamely ramified $\Omega(L/K, 2)$ is equal to the class of the ring of integers, $\mathcal{O}_L$. The Fröhlich-Chinburg conjecture (Conjecture 2.2.6) asserts that $\Omega(L/K, 2) = W_{L/K}$. In Section 2.3, using p-adic L-functions, the Hom-description representative is computed for $\Omega(\mathbf{Q}^+(\xi_{p^{s+1}})/\mathbf{Q}, 2)$ when p is an odd, regular prime and $\mathbf{Q}^+(\xi_{p^{s+1}})/\mathbf{Q}$ is the totally real, p-primary cyclotomic extension. As a consequence, in Theorem 2.3.12, we verify the Fröhlich-Chinburg conjecture for the intermediate extension, $F_{s+1}/\mathbf{Q}$, of degree p^s.

In Chapter Three we develop logarithmic techniques for detecting elements of $\mathcal{CL}(\mathbf{Z}[G])$. In Section 3.1 we use Explicit Brauer Induction to derive some determinantal congruences. These are congruences satisfied by functions on $R(G)$ which are given by determinants of units in local group-rings. The first application of these congruences is to give a simple construction of the group-ring logarithm, L_G, of R. Oliver and M.J. Taylor. The role of L_G is to transform the complicated multiplicative functions which appear in the Hom-description into simpler additive functions. As an application, in Section 3.2, we establish the tame Galois descent for determinantal functions (Theorem 3.2.5) which is one of the principal ingredients in M.J. Taylor's proof of Fröhlich's conjecture (see Theorem 4.1.5). Another consequence of the determinantal congruences of Section 3.1 is that the logarithm of a determinantal function also satisfies a number of congruences. In Section 3.3, at odd primes, we prove the converse (Theorem 3.3.21)—namely that almost any function satisfying the congruences is the logarithm of a determinant. Since determinantal functions are trivial in the class-group, this gives a criterion for the vanishing of classes in $D(\mathbf{Z}[G])$ when G is a p-group of odd order (Theorem 3.3.28).

In Chapter Four we consider the case when L/K is a tamely ramified Galois extension of number fields. In this case $\Omega(L/K, 2) = [\mathcal{O}_L]$, the class of the integers of L. In this case Fröhlich conjectured that $[\mathcal{O}_L] = W_{L/K}$. In Section 4.1 we describe Fröhlich's Hom-description representative for $[\mathcal{O}_L]$ and sketch M.J. Taylor's proof of Fröhlich's conjecture, culminating (Theorem 4.1.5) in the use of tame Galois descent for determinantal functions. In Section 4.2 we develop a new approach, using the congruence results of Theorem 3.3.28 to prove (Theorem 4.2.8) M.J. Taylor's result in the special case of a p-group of odd order.

In Chapter Five we consider the class-group of a maximal order of the rational group-ring. In Section 5.1 we describe a family of Hom-groups associated to cyclic subquotients of G and the manner in which Explicit Brauer Induction may be used to construct a split injection of the class-group of the maximal order into the inverse limit of this family (Theorem 5.1.9). In Section 5.2 the method of canonical factorisations is explained. This is a method by which to obtain Hom-description representatives of classes in the class-group of the maximal order. In Section 5.3 canonical factorisation is used to give a proof of D. Holland's result that the Fröhlich-Chinburg conjecture holds in the class-group of the maximal order. We give a fairly complete proof of this result (Theorem 5.3.9) in the case when all the wild decomposition groups are cyclic (the locally cyclic case). This special case adequately illustrates how canonical factorisations are manipulated and may be amplified into a proof of the general case (see Snaith [1994], Section 7.3) using the results of Section 5.1.

In Chapter Six we give an alternative method of some quaternionic cases of the Fröhlich-Chinburg conjecture which were first studied by S. Kim. These examples concern Galois extensions, $N/\mathbf{Q}$, whose group is isomorphic to the quaternion group of order eight which is also the decomposition group at $p = 2$. In Section 6.1 we examine the fundamental 2-adic local 2-extension and obtain an injective chain map of the standard Q_8-resolution into it. It is in this first step that our method departs from that of Kim [1992]. In Section 6.2 we use cohomological techniques to evaluate the (cohomologically trivial) cokernel of the injection of Section 6.1. In Section 6.3 we complete the computation by evaluating $\Omega(N/\mathbf{Q}, 2)$ in comparison to $W_{N/\mathbf{Q}}$.

In Chapter Seven we construct some new class-group invariants of the Chinburg type. These are made from a cohomological study of the higher algebraic K-groups of rings of S-integers. At the moment I have very little information concerning these new classes but, as is explained in Remark 7.1.5, these new classes and their Hom-description representatives should be part of a class-group reflection of the Lichtenbaum and Stark conjectures.

In conclusion I should like to thank Carolyn, Nina and Daniel—for tolerating my nocturnal typing—and my students, Jeff Hooper and Minh Tran—for scrutinising the example in Chapter Six—and The Fields Institute—for offering me the opportunity to give these lectures.

Victor Snaith,
McMaster University,
April 1994.

This page intentionally left blank

Index

This page intentionally left blank

Bibliography

Boltje, R. [1989] *Canonical and explicit Brauer induction in the character ring of a finite group and a generalisation for Mackey functors*, Augsburg Univ. thesis.

——— [1990] *A canonical Brauer induction formula*, Astérisque, **t.181-182**, 31–59.

Boltje, R., Snaith, V. and Symonds, P. [1992] *Algebraicisation of Explicit Brauer Induction*, J. Alg. **(2) 148**, 504–527.

Brauer, R. [1951] *Bezeihungen zwischen Klassenzahlen von Teilkörpern eines Galoischen Körpers*, Math. Nachr., **4**, 158–174.

Burns, D. [1991] *Canonical factorisability and a variant of Martinet's conjecture*, J. London Math. Soc., **142**, 24–46.

Cassou-Noguès, Ph. [1978] *Quelques théorèmes de base normal d'entiers*, Ann. Inst. Fourier, **28**, 1–33.

Cassou-Noguès, Ph., Chinburg, T., Fröhlich, A. and Taylor, M.J. [1989] *L-functions and Galois modules*, London Math. Soc. Durham Symposium, L-functions and Arithmetic, Cambridge University Press, 75–139.

Chinburg, T. [1983] *On the Galois structure of algebraic integers and S-units*, Inventiones Math., **74**, 321–349.

——— [1985] *Exact sequences and Galois module structure*, Annals Math., **121**, 351–376.

——— [1989] *The Analytic Theory of Multiplicative Galois Structure*, Mem. A.M. Soc., **395**.

Curtis, C.W. and Reiner, I. [1981,1987] *Methods of Representation Theory*, **Vols. I & II**, Wiley.

Deligne, P. and Ribet, K. [1980] *Values of abelian L-functions at negative integers over totally real fields*, Invent. Math. **59**, 227–286.

Greenberg, R. [1983] *On p-adic Artin L-functions*, Nagoya Math. J., **89**, 77–87.

Fröhlich, A. [1976] *Arithmetic and Galois module structure for tame extensions*, J. Reine. Angew. Math., **286/7**, 380–440.

——— [1983] *Galois Module Structure of Algebraic Integers* , Ergeb. Math., **3**, Folge-Band 1, Springer-Verlag.

——— [1984] *Classgroups and Hermitian Modules*, Progress in Math., **48**, Birkhäuser.

——— [1988] *Module defect and factorisability*, Ill. J. Math., **32**, 407–421.

Fröhlich, A. and Queyrut, J. [1973] *On the functional equation for Artin L-functions for characters of real representations*, Inv. Math. **20**, 125–138.

Henniart, G. [1984] *Galois ϵ-factors modulo roots of unity*, Invent. Math., **78**, 117–126.

Hilton, P.J. and Stammbach, U. [1971] *A Course in Homological Algebra*, Grad. Texts in Math., **4**, Springer-Verlag.

Holland, D. [1992] *Additive Galois module structure and Chinburg's invariant*, J. reine angew. Math, **425**, 193–218.

Iwasawa, K. [1986] *Local Class Field Theory* , Oxford Math. Monographs, Oxford University Press.

Jensen, C.U. and Yui, N. [1987] *Quaternion extensions*, Conf. Alg. Geom. and Comm. Alg. in honour of M. Nagata, 155–182.

Kahn, B. [1993] *Descente Galoisienne et K_2 des corps de nombres*, K-theory,**7 (1)**, 55–100.

Kato, K. and Saito, S. [1983] *Two dimensional class field theory*, Galois Groups and their Representations, Proc. Symp. Nagoya 1981, Advanced Studies in Pure Maths., **2**, 103–152.

Kim, S. [1991] *A generalization of Fröhlich's theorem to wildly ramified quaternion extensions of* $\mathbf{Q}$, Ill. J. Math., **35**, 158–189.

——[1992] *The root number class and Chinburg's second invariant*, J. Alg. **153**, 133–202.

Lang, S. [1970] *Algebraic Number Theory*, Addison-Wesley.

Lang, S. [1984] *Algebra*, 2nd ed., Addison-Wesley.

Levine, M. [1989] *The indecomposable K_3 of a field*, Ann. Sci. École Norm. Sup., **22**, 255–344.

Lichtenbaum, S. [1973] *Values of zeta functions, étale cohomology and algebraic K-theory*, Lecture Notes in Maths., **342**, 489–501.

Martinet, J. [1977a] H_8, London Math. Soc. 1975 Durham Symposium, Algebraic Number Fields, Academic Press, 525–538.

——[1977b] *Character theory and Artin L-functions*, London Math. Soc., 1975 Durham Symposium, Algebraic Number Fields, Academic Press, 1–87.

Milnor, J.W. [1970] *Algebraic K-theory and quadratic forms*, Inventiones Math., **9**, 318–344.

——[1971] *Introduction to Algebraic K-theory*, Ann. Math. Studies, **72**, Princeton University Press.

Merkurjev, A.S. and Suslin, A.A. [1983] *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Izv. Nauk. SSSR 46 (1982), 1011–1046 (Eng. trans., Math. USSR 21, 307–340).

——[1990] *The K_3 group of a field*, Izv. Nauk. SSSR **54** (1990) 339–356 (Eng. trans., Math. USSR, **36**, 541–565).

Neukirch, J. [1988] *The Beilinson conjecture for algebraic fields*, Birkhauser Prospects in Math, **4**, 193–247.

Noether, E. [1932] *Normalbasis bei Körpern ohne höhere Verzweigung*, J. reine angew. Math., **167**, 147–152.

Oliver, R. [1980] *SK_1 for finite group rings II*, Math. Scand., **47**, 195–231.

Quillen, D.G. [1972] *On the cohomology and K-theory of the general linear groups over a finite field*, Annals of Math., **96**, 552–586.

——— [1973] *Higher Algebraic K-theory I*, Algebraic K-theory I, 85–147, Lecture Notes in Math., **341**, Springer-Verlag.

Serre, J-P. [1977] *Linear Representations of Finite Groups*, Grad. Texts in Math., **42**, Springer-Verlag.

——— [1979] *Local Fields*, Grad. Texts in Math., **67**, Springer-Verlag.

Snaith, V.P. [1988] *Explicit Brauer Induction*, Inventiones Math., **94**, 455–478.

——— [1989a] *Topological Methods in Galois Representation Theory*, C.M. Soc. Monographs, Wiley.

——— [1989b] *Invariants of Representations*, NATO ASI Series # 279, Algebraic K-theory: Connections with Geometry and Topology, 445–508, Kluwer.

——— [1994] *Explicit Brauer Induction (with applications to algebra and number theory)*, Cambridge Studies in Advanced Math., Cambridge University Press.

Spanier, E.H. [1966] *Algebraic Topology*, McGraw-Hill.

——— [1984] *On the K-theory of local fields*, J. Pure Appl. Algebram, **34**, 301–318.

Suslin, A.A. [1987] *Torsion in K_2 of fields*, K-theory 1, 5–29.

Symonds, P. [1991] *A splitting principle for group representations*, Comm. Math. Helv., **66**, 169–184.

Tate, J.T. [1976] *Relations between K_2 and Galois cohomology*, Inventiones Math., **36**, 257–274.

——— [1977] *Local Constants*, London Math. Soc., 1975 Durham Symposium, Algebraic Number Fields, (A. Fröhlich, ed.), Academic Press, 89–131.

——— [1984] *Les Conjectures de Stark sur les Fonctions L d'Artin en $s = 0$*, Progress in Mathematics, **47**, Birkhauser.

Taylor, M.J. [1978a] *Locally free classgroups of groups of prime power order*, J. Algebra, **(2) 50**, 463–487.

——— [1978b] *Galois module structure of integers of relative abelian extensions,* J.Reine. Angew. Math., **303, 4**, 97–101.

——— [1979] *Adams operations, local root numbers and the Galois module structure of rings of integers*, Proc. London Math. Soc., **(3) 39**, 147–175.

——— [1980] *A logarithmic approach to class groups of integral group rings*, J. Algebra, **66**, 321–353.

——— [1981] *On Fröhlich's conjecture for rings of integers of tame extensions* , Inventiones Math., **63**, 41–79.

——— [1984] *Class Groups of Group Rings*, L.M. Soc. Lecture Notes Series, **91**, Cambridge University Press.

Wall, C.T.C. [1979] *Periodic projective resolutions*, Proc. London Math. Soc., **3**, 509–553.

Washington, L. [1982] *Introduction to Cyclotomic Fields*, Grad. Texts in Math. **83**, Springer-Verlag.

Wiles, A. [1990] *The Iwasawa conjecture for totally real fields*, Annals of Math., **131**, 493–540.

Wilson, S.M.J. [1990] *A projective invariant comparing integers in wildly ramified extensions*, J. reine angew. Math., **412**, 35–47.

Galois Module Structure

Victor P. Snaith

Galois module structure deals with the construction of algebraic invariants from a Galois extension of number fields with group G. Typically these invariants lie in the class-group of some group-ring of G or of a related order. These class-groups have "Hom-descriptions" in terms of idèlic-valued functions on the complex representations of G. Following a theme pioneered by A. Frölich, T. Chinburg constructed several invariants whose Hom-descriptions are (conjecturally) given in terms of Artin root numbers. For a tame extension, the second Chinburg invariant is given by the ring of integers, and M. J. Taylor proved the conjecture in this case. The first published graduate course on the Chinburg conjectures, this book provides the necessary background in algebraic and analytic number theory, cohomology, representation theory, and Hom-descriptions. The computation of Hom-descriptions is facilitated by Snaith's Explicit Brauer Induction technique in representation theory. In this way, illustrative special cases of the main results and new examples of the conjectures are proved and amplified by numerous exercises and research problems. The final chapter introduces a new invariant constructed from algebraic K-theory, whose Hom-description is related to the L-function value at $s = -1$.

ISBN 0-8218-0264-X

9 780821 802649