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Spencer J. Bloch

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The Centre de Recherches Mathématiques (CRM) of the Université de Montréal was created in 1968 to promote research in pure and applied mathematics and related disciplines. Among its activities are special theme years, summer schools, workshops, postdoctoral programs, and publishing. The CRM is supported by the Université de Montréal, the Province of Québec (FQRNT), and the Natural Sciences and Engineering Research Council of Canada. It is affiliated with the Institut des Sciences Mathématiques (ISM) of Montréal, whose constituent members are Concordia University, McGill University, the Université de Montréal, the Université du Québec à Montréal, and the Ecole Polytechnique.

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Abstract. Work of Borel on higher regulators for number fields is discussed. The Borel regulator for K_3 of a number field is described explicitly in terms of the dilogarithm function.

A generalization, based on functions related to the dilogarithm and to the Dedekind η-function, leads to a regulator for K_2 of an elliptic curve E over a number field. Elements in $K_2(E)$ analogous to cyclotomic units are described. The regulator is evaluated on these elements and the resulting values related to the value of the Hasse-Weil zeta function of E at $s = 2$ when E has complex multiplication. This regulator formula is worked out in detail for the case of E defined over $\mathbb{Q}$ with complex multiplication by the ring of integers in an imaginary quadratic field, when it takes a particularly simple form.

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Author's Apology

"The author apologizes for the long delay in publishing this monograph. The reader should understand that since this work was done, fundamental ideas of A. Beilinson, A. Suslin, V. Voevodsky, and others have totally transformed the landscape. Sometimes it is fun to drive around in a Model T Ford but one should be aware there are much faster cars on the road."

Contents

Acknowledgments

I should like to express my gratitude to Bill Messing and the Department of Mathematics at Irvine for inviting me to give these lectures. At the time of the lectures the whole relationship between the regulator and the zeta function of an elliptic curve was purely conjectural. Messing's interest gave me the fortitude to push through the complicated business. I also want to acknowledge many fruitful conversations with D. Wigner about the dilogarithm function. It was he who first wrote down the function $D(x)$ (0.3.4) and worked out a number of its properties.

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This book is the long-awaited publication of the famous Irvine lectures by Spencer Bloch. Delivered in 1978 at the University of California at Irvine, these lectures turned out to be an entry point to several intimately-connected new branches of arithmetic algebraic geometry, such as regulators and special values of L-functions of algebraic varieties, explicit formulas for them in terms of polylogarithms, the theory of algebraic cycles, and eventually the general theory of mixed motives which unifies and underlies all of the above (and much more). In the 20 years since, the importance of Bloch's lectures has not diminished. A lucky group of people working in the above areas had the good fortune to possess a copy of old typewritten notes of these lectures. Now everyone can have their own copy of this classic.

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