CONTEMPORARY MATHEMATICS

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Ramanujan 125

International Conference to Commemorate the
125th Anniversary of Ramanujan's Birth
Ramanujan 125
November 5–7, 2012
University of Florida, Gainesville, Florida

Krishnaswami Alladi
Frank Garvan
Ae Ja Yee
Editors

American Mathematical Society

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Photograph of Participants, Ramanujan 125 Conference, University of Florida, November 5–7, 2012. Courtesy of Ali Uncu.

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Preface

An international conference "Ramanujan 125" was held November 5–7, 2012, in Gainesville, Florida. The conference, which was organized by Krishnaswami Alladi and Frank Garvan of the University of Florida, and Ae Ja Yee of The Pennsylvania State University, attracted 70 active research participants from around the world. The conference was supported by grants from the National Science Foundation and the National Security Agency and by funds from The Pennsylvania State University through the NSF grant of Ae Ja Yee. We are most grateful for this support that was crucial to the success of the conference.

The conference featured ten plenary talks of one hour each by leaders in the world of Ramanujan's mathematics, and 40 shorter presentations including several by graduate students. These lectures discussed significant progress in various branches of mathematics in the quarter century since Ramanujan's centennial — progress directly related to Ramanujan's work or topics whose origins can be traced to Ramanujan's discoveries. This *Contemporary Mathematics* volume is the refereed proceedings of the conference and contains research and expository papers based on talks delivered at the conference. All papers have been arranged in alphabetical order of the first author's last name.

In his last letter to Hardy in January 1920, Ramanujan communicated his discovery of the mock theta functions, which mimic the theta functions in the sense that their coefficients can be estimated to the same degree of precision as in the case of objects expressible in terms of theta functions. The mock theta functions are now considered to be among Ramanujan's deepest contributions. Ramanujan had obtained asymptotic evaluations of these mock theta functions, and in his letter had observed that if certain well-behaved analytic expressions were subtracted from the mock theta functions, then the resulting error would be bounded. He also indicated bounds in certain instances. For many years the exact links between mock theta functions and modular forms were unknown, and this was one of many such tantalizing mysteries.

In the last decade, Ken Ono, Kathrin Bringmann, and their collaborators have connected mock theta functions to harmonic Maass forms, thereby providing the key to unlock this mystery by developing the ideas in a fundamental 2003 PhD thesis of Sander Zwegers that was written under the direction of Don Zagier in Bonn.

On the opening day of this conference, Ono announced for the first time his recent work with Amanda Folsom and Robert Rhoades, in which they obtain a precise expression for the bounded error Ramanujan indicated. We are pleased that the three authors have contributed a paper on this topic to this *Contemporary Mathematics* volume.

Ramanujan's discovery of congruences for the partition function startled Hardy who never expected divisibility properties for objects defined by an additive process. Over the last century, Ramanujan's congruences have inspired the more general study of congruence properties satisfied by coefficients of modular forms. Following the lead of Hecke, Oliver Atkin obtained new types of such congruences. The paper by Ahlgren and Andersen on Hecke grids and congruences for weakly holomorphic modular forms provides certain infinite families of weakly holomorphic forms, and establishes stronger versions of various congruences conjectured by Honda and Kaneko using the Atkin operator.

Ramanujan is considered the all time greatest in the area of continued fractions. In one of his letters to Hardy in 1913, Ramanujan provided astounding evaluations of what is now called the Rogers–Ramanujan continued fraction, and claimed a lovely modular identity connecting the values of this fraction at arguments x and x^5. It was results like these that convinced Hardy and his Cambridge colleagues that Ramanujan was in the league of Euler and Jacobi. For this reason, Michael Hirschhorn notes that this modular identity for the Rogers–Ramanujan continued fraction changed the course of history, and in his paper provides a much simpler proof of this gem of Ramanujan. The semi-expository paper of Gaurav Bhatnagar provides an elegant unified approach to several continued fraction of Ramanujan, including the Rogers–Ramanujan continued fraction. To supplement these two lovely papers on continued fractions, Lisa Lorentzen, an authority in this area, has contributed a paper rigorously studying the convergence of random continued fractions.

In connection with the two Rogers–Ramanujan identities, the late Leon Ehrenpreis had asked whether one could prove that the coefficients in the first identity are larger than those of the second by not relying on the series forms but by just considering the congruential product representation. This was settled recently by Kevin Kadell. Motivated by this, Alex Berkovich and Keith Grizzell have taken up a systematic study more generally of finite products, and in their paper provide intricate injective maps to establish such positivity results.

George Andrews, the leader in the theory of partitions and q-series, and on Ramanujan's work, has been responsible for opening up several new directions of investigation in these fields motivated by surprising new connections between partitions, q-series, and other areas of mathematics. Garoufaldis, Le, and Zagier discovered some remarkable q-series identities in the course of studying certain knots. In his paper, Andrews has generalized and refined these identities by introducing free parameters; by proving them using q-series, he has explained their place in the hierarchical structure of q-hypergeometric identities.

During the Ramanujan Centennial, Bill Gosper humorously remarked that Ramanujan reaches out from his grave and snatches away your best formulas. That is, Ramanujan had anticipated much important work by later authors. A good example is an important formula of Koshliakov of 1929 which can actually be found in Ramanujan's Lost Notebook. The Koshliakov formula is a functional equation for a certain Bessel function and involves the divisor function as well. Just as the theta transformation formula is related to the Riemann zeta function $\zeta(s)$, Koshliakov's formula is equivalent to a functional equation for $\zeta^2(s)$. Bruce Berndt, Sun Kim, and Alexander Zaharescu provide new analogues to Koshliakov's formula by

considering character analogues of the divisor function. They indicate applications to the positivity of the values $L(1, \chi)$ of L-functions involving even characters χ.

The study of Lie algebras has led to the discovery of new Rogers–Ramanujan type identities. Kailash Misra and Evan Wilson study tensor product decompositions of certain modules arising in the investigation of infinite dimensional Kac–Moody algebras, and deduce some new partition identities as a consequence. Similarly, the study of theta constants has also led to the discovery of new partition identities by Farkas and Kra in the last decade. Here, Herschel Farkas, J.Y. Kaminski and E. Yakubov, lay the foundations of a theory of theta constant identities for certain non-singular curves, and indicate applications.

Highly composite numbers were first studied by Ramanujan. In the course of that study, Ramanujan considered also the maximal order of the sum of divisors function, and noted that the Riemann Hypothesis implied a precise estimate for its maximal order. In recent years, Nicolas and Robin have investigated the results in an unpublished second manuscript of Ramanujan on highly composite numbers. Here Jean-Louis Nicolas and Jonathan Sondow describe the fascinating history surrounding highly composite numbers and discuss important contributions by Ramanujan and later authors.

The celebrated theorem of Lagrange that every positive integer is a sum of four squares motivated Ramanujan to write down 51 examples of universal quadratic forms, namely those that represent all positive integers. It is also of interest to determine the number of representations and express these in terms of sums of well-known multiplicative functions. Cherng-tiao Perng studies two quaternary quadratic forms and their number of representations and proves his results using the algebra of quaternions.

Finally, the paper of Christian Krattenthaler and Michael Schlosser deals with the study of the major index generating function of Young tableaux, and uses techniques from the theory of multi-dimensional q-hypergeometric series, as well as transformation formulas for elliptic hypergeometric series. Thus, the volume contains a collection of papers spanning a broad spectrum of mathematics and representing areas where Ramanujan has had, and continues to have, a major influence.

We appreciate both the support of Ed Dunne and Christine Thivierge of the AMS, and their interest in having these proceedings as a part of the AMS *Contemporary Mathematics* series. We are also thankful to Ms. Margaret Somers for efficiently handling the needs of the conference participants, as well as all of the local arrangements for the conference. We thank Ali Uncu for the conference group photo.

Krishnaswami Alladi, Gainesville, Florida,
Frank Garvan, Gainesville, Florida, and
Ae Ja Yee, State College, Pennsylvania
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This volume contains the proceedings of an international conference to commemorate the 125th anniversary of Ramanujan's birth, held from November 5–7, 2012, at the University of Florida, Gainesville, Florida.

Srinivasa Ramanujan was India's most famous mathematician. This volume contains research and survey papers describing recent and current developments in the areas of mathematics influenced by Ramanujan. The topics covered include modular forms, mock theta functions and harmonic Maass forms, continued fractions, partition inequalities, q-series, representations of affine Lie algebras and partition identities, highly composite numbers, analytic number theory and quadratic forms.

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