CONTEMPORARY MATHEMATICS

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Mathematical Aspects of Quantization

Center for Mathematics at Notre Dame

Summer School and Conference
May 31–June 10, 2011
Notre Dame University, Notre Dame, Indiana

Sam Evens
Michael Gekhtman
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We dedicate this volume in memory of Jean-Marie Souriau, in honor of his contributions to quantization.

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Preface

Quantization is an important topic in mathematics and physics. From the physics point of view, methods of quantization are procedures for building models for quantum mechanical systems from analogous and more intuitive classical mechanical systems, which provide strikingly precise experimental predictions. Much of the development of theoretical physics in the 20th century may be regarded as the process of refining quantization to give improved experimental predictions, and the search for a unified field theory is an attempt to quantize general relativity in a manner compatible with existing quantum theory. On the mathematics side, problems related to quantization and quantum mechanics was a strong motivation for the development of functional analysis, the representation theory of Lie groups, and spectral geometry. More recent developments with much current activity include geometric quantization, deformation quantization, and quantum analogues of various classical objects.

Geometric quantization seeks to give as natural as possible a procedure for associating a Hilbert space to each symplectic manifold satisfying an integrality condition. Geometric quantization was developed by Kirillov, Kostant, and Souriau in the 1960's, and was energized by the attempt to prove the "quantization commutes with reduction" conjecture of Guillemin and Sternberg in the 1980's and 1990's, which was completed in part by work of Meinrenken, and is continued through the study of its L^2-analogues. Much work has been done on the semiclassical asymptotics of geometric quantization, such as the proof by Bordemann, Meinrenken, and Schlichenmaier of a general asymptotic formula for Berezin–Toeplitz quantization on compact Kähler manifolds. Semiclassical analysis has resulted in applications in number theory, as in the work of Borthwick and Uribe on relative Poincare series.

The theory of deformation quantization seeks to deform the commutative algebra of functions on a Poisson manifold into a noncommutative algebra in which the semi-classical limit is given by the Poisson bracket of functions. Kontsevich's proof of his formality conjecture showed that every Poisson manifold has a star product on the formal level, and this work was one of the key results which earned him the Fields medal. This work was later reinterpreted by Tamarkin and related to path integrals by Cattaneo and Felder. Ideas from deformation quantization also play a central role in recent work of Costello giving a rigorous geometric construction of the Witten genus. Deformation quantization is used by Etingof and Ginzburg to give a better geometric understanding of the rational Cherednik algebra and its representations, as well as for other associative algebras.

The papers in this volume are based on talks given at the Center for Mathematics at Notre Dame program on quantization, which was held from May 31 to June 10 of 2011. The program consisted of a summer school on quantization, followed by

a conference titled "Mathematical aspects of quantization". The papers by Berest–Samuelson, Dolgushev–Rogers, Lerman, and Meinrenken are based on talks given at the summer school. The paper by Berest and Samuelson begins with an elegant proof of properties of the Dunkl operators using a deformation of the de Rham complex, and continues to discuss the representation theory of the Cherednik algebra. The paper by Dolgushev and Rogers is concerned with the graph complex, which plays a key role in the Kontsevich formality conjecture, and results of Willwacher which relate the cohomology of the graph complex to the cohomology of the Gerstenhaber operad. Dolgushev and Rogers give a detailed and complete discussion of Willwacher's proof and the necessary background. The paper by Lerman gives a short introduction to geometric quantization, which we hope will make the subject more accessible to graduate students. The paper by Meinrenken gives a survey of the theory of group-valued moment maps and its applications to the moduli space of flat bundles on a surface, which was developed by Meinrenken together with his collaborators Alekseev, Malkin, and Woodward. The remaining papers are based on talks at the conference. Barron discusses interactions between quantization and automorphic forms. Berest, Chen, Eshmatov, and Ramadoss discuss derived versions of Poisson structures and their applications to Calabi–Yau algebras. Kar and Rajeev give an elementary explanation of renormalization. Schlichenmaier's paper gives a survey of Berezin–Toeplitz quantization and star products in the Kähler setting. Śniatycki gives a survey of his results concerning commutation of geometric quantization with algebraic reduction.

Jean-Marie Souriau, who was one of the pioneers in the theory of quantization, passed away on March 15, 2012, as this volume was being prepared. Souriau's 1966 paper, "Quantification géométrique," in Communications on Mathematical Physics was one of the seminal papers leading to the modern theory of geometric quantization. Souriau also made important contributions to the study of moment maps and coadjoint orbits, both of which are by now standard tools in the quantization toolbox. Souriau spent most of his career as Professor of Mathematics at the University of Provence in Marseille.

The Center for Mathematics at Notre Dame provided the resources to run the program in quantization, and we would like to thank Gregory Crawford, Dean of the College of Science at Notre Dame, for enabling us to establish this new center. We would also like to thank our colleagues at Notre Dame for helping organize the center, and especially thank Lisa Tranberg for her efficient organization of the conference. Finally, we would like to thank all of the participants in our quantization program for making the program such an interesting event.

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Mathematical Aspects of Quantization • Evens et al., Editors

This book is a collection of expository articles from the Center for Mathematics at Notre Dame's 2011 program on quantization.

Included are lecture notes from a summer school on quantization on topics such as the Cherednik algebra, geometric quantization, detailed proofs of Willwacher's results on the Kontsevich graph complex, and group-valued moment maps.

This book also includes expository articles on quantization and automorphic forms, renormalization, Berezin-Toeplitz quantization in the complex setting, and the commutation of quantization with reduction, as well as an original article on derived Poisson brackets.

The primary goal of this volume is to make topics in quantization more accessible to graduate students and researchers.

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