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Highest weight modules at the critical level and noncommutative Springer resolution

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Abstract. In the article by Bezrukavnikov and Mirković a certain non-commutative algebra A was defined starting from a semi-simple algebraic group, so that the derived category of A-modules is equivalent to the derived category of coherent sheaves on the Springer (or Grothendieck-Springer) resolution.

Let $\mathfrak{g}^\vee$ be the Langlands dual Lie algebra and let $\widehat{\mathfrak{g}}$ be the corresponding affine Lie algebra, i.e. $\widehat{\mathfrak{g}}$ is a central extension of $\mathfrak{g}^\vee \otimes \mathbb{C}((t))$.

Using results of Frenkel and Gaitsgory we show that the category of $\widehat{\mathfrak{g}}$ modules at the critical level which are Iwahori integrable and have a fixed central character, is equivalent to the category of modules over a central reduction of A. This implies that numerics of Iwahori integrable modules at the critical level is governed by the canonical basis in the K-group of a Springer fiber, which was conjecturally described by Lusztig and constructed by Bezrukavnikov and Mirković.

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1. Introduction

1.1. Modules at the critical level. Category O of highest weight modules over a semi-simple Lie algebra with a fixed central character is a classical object of study in representation theory; Kazhdan-Lusztig conjectures (proved by Beilinson–Bernstein and Brylinksi–Kashiwara) assert that numerics of such modules is governed by the canonical basis in the Hecke algebra. The subject of this paper is an analogue of that result for modules over an affine Lie algebra at the critical level. We show that the category of such modules is governed by the canonical bases in the Grothendieck group (or homology) of Springer fibers. This basis was described conjecturally by Lusztig [**14**] and its existence was established in [**5**]. The original motivation for [**5**] came from representation theory of Lie algebras in positive

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characteristic; it turns out that the same generalization of Kazhdan-Lusztig theory controls highest weight modules at the critical level.

Let G be a semi-simple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra corresponding to the *Langlands dual* Lie algebra $\check{\mathfrak{g}}$. Thus $\widehat{\mathfrak{g}}$ is a central extension of the loop algebra

$$0 \to \mathbb{C}^r \to \widehat{\mathfrak{g}} \to \check{\mathfrak{g}} \otimes \mathbb{C}((t)) \to 0,$$

where r is the number of simple summands in $\mathfrak{g}$.

Let $U_{crit}\widehat{\mathfrak{g}}$ denote the quotient of the enveloping algebra $U\widehat{\mathfrak{g}}$ at the critical value of the central charge.

Let $U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}$ denote the category of Iwahori monodromic $U_{crit}\widehat{\mathfrak{g}}$ modules in the sense of [**10**]. Recall that by the result of Feigin and Frenkel [**8**] a continuous $U_{crit}\widehat{\mathfrak{g}}$ module carries a canonical commuting action of the topological ring $\mathcal{O}(Op)$ of functions on the space Op of G-opers on the formal punctured disc. In particular, an irreducible module has a central character which corresponds to such an oper. For an irreducible module $L \in U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}$ the oper necessarily has a regular singularity and a nilpotent residue.

Fix a nilpotent element $e \in \mathfrak{g}$ and a nilpotent oper $\mathfrak{O}$ with residue e (thus, the underlying connection is isomorphic to $\nabla = d + e\frac{dt}{t}$ where t is a coordinate on the formal disc). We let $U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}_{\mathfrak{O}}$ be the full subcategory in $U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}$ consisting of finite length modules where $\mathcal{O}(Op)$ acts through the character corresponding to $\mathfrak{O}$.

1.2. Noncommutative Springer resolution. We now introduce another abelian category associated to the nilpotent element e. Let $\mathcal{B} = G/B$ be the flag variety of G thought of as the variety of Borel subalgebras in $\mathfrak{g}$; let $\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \mid \mathfrak{b} \in \mathcal{B}, x \in \mathfrak{b}\} \xrightarrow{\pi} \mathfrak{g}$ be the Grothendieck-Springer map $\pi : (\mathfrak{b}, x) \mapsto x$.

In [**5**] a certain non-commutative algebra A, well defined up to a Morita equivalence was introduced. The algebra comes equipped with an equivalence of triangulated categories $D^b(A\text{–mod}^{fg}) \cong D^b(Coh(\tilde{\mathfrak{g}}))$; by $A\text{–mod}^{fg}$ we denote the category of finitely generated A-modules.

The center of A is identified with the algebra $\mathcal{O}(\mathfrak{g})$ of regular functions on $\mathfrak{g}$. For $e \in \mathfrak{g}$ we let A_e denote the corresponding central reduction of A.

The results of [**5**] provide a canonical isomorphism of Grothendieck groups $K^0(A_e\text{–mod}^{fg}) \cong K^0(Coh(\pi^{-1}(e)))$ sending the classes of irreducible modules to elements of the *canonical basis*, i.e. the unique (up to signs) basis satisfying the axioms of [**14**].

The next statement conjectured in [**5**, Conjecture 1.7.2] is the main result of the present note.

THEOREM 1. *There exists a canonical equivalence of abelian categories*

$$A_e\text{–mod}^{fg} \cong U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}_{\mathfrak{O}}.$$

In fact, the equivalence of derived categories follows by comparing the result of [**9**] which identifies $D^b(U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}_{\mathfrak{O}})$ with $DGCoh(\mathcal{B}_e)$ with that of [**5**], [**16**] which identify $D^b(A_e\text{–mod}^{fg})$ with the same category of coherent sheaves. Here $DGCoh(\mathcal{B}_e)$ denotes the derived category of coherent sheaves on the DG-scheme $\mathcal{B}_e = \{e\} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \tilde{\mathfrak{g}}$. (Notice that $K^0(DGCoh(\mathcal{B}_e)) = K^0(Coh(\pi^{-1}(e)))$ since

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the Grothendieck group of coherent complexes on a DG-scheme is identified with the Grothendieck group of coherent sheaves on the underlying scheme.)

Our job in the present note is to show that the resulting equivalence

$$D^b(A_e\text{–mod}^{fg}) \cong D^b(U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}_{\mathcal{O}})$$

induces an equivalence of abelian categories, i.e. that it is t-exact with respect to the natural t-structures. This will be done using characterizations of the t-structure on $DGCoh(\mathcal{B}_e)$ coming from the two equivalences with derived categories of modules, appearing, respectively, in [**5**], [**9**].

According to [**5**, Conjecture 1.7.1], the category of A_e–mod^{fg} is equivalent to a category of modules over the Kac - De Concini quantum group at a root of unity. Thus, together with the present result, that Conjecture implies an equivalence between modules over the affine Lie algebra and quantum group modules at a root of unity. Another equivalence of this sort has been established in the celebrated work by Kazhdan and Lusztig [**13**].

We also expect that when the nilpotent e is of principal Levi type (a generalization of) our result can be used to derive character formulas for irreducible highest weight modules in terms of parabolic periodic Kazhdan-Lusztig polynomials; we plan to develop this application in a future work.

The rest of the text is structured as follows. In section 2 we recall the needed properties of the noncommutative Springer resolution including a characterization of the corresponding t-structure on the derived categories of coherent sheaves. Section 3 is devoted to constructible sheaves on affine flag variety of the dual group. We state a description of the subcategory of complexes equivariant with respect to the radical of the Iwahori subgroup I^0 in terms of coherent sheaves on Steinberg variety of G, to appear in [**4**]. A technical result about the t-structure on the category of Iwahori-Whittaker sheaves appearing in Proposition 1 is the key statement providing a link between the description of the t-structure by Frenkel-Gaitsgory [**9**] to our formalism of braid positive t-structures. In section 4 we quote the result of [**9**] and argue, in subsection 4.2, that the two characterizations are compatible, which yields Theorem 1.

1.3. Conventions and notations. Let W_{aff} denote the semi-direct product of the Weyl group W by the weight lattice Λ of G. Thus W_{aff} is an extended affine Weyl group corresponding (in the Bourbaki terminology) to the *dual group* $G\check{}$. Let ℓ denote the length function on W_{aff}. Let B_{aff} be the corresponding extended affine braid group and $B^+_{aff} \subset B_{aff}$ be the semigroup of positive braids, i.e. the semigroup consisting of products of the Coxeter generators (but not their inverses) and length zero elements. Thus B_{aff} surjects onto W_{aff}. We have a section of the map $B_{aff} \to W_{aff}$ sending an element $w \in W_{aff}$ to its minimal length preimage $\tilde{w} \in B^+_{aff}$. The elements $\tilde{w}$ generate B_{aff} subject to the relation $\widetilde{w_1 w_2} = \tilde{w}_1\tilde{w}_2$ provided that $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$. Let Σ (respectively, Σ_{aff}) be the set of vertices of Dynkin diagram (respectively, affine Dynkin diagram) of $\mathfrak{g}\check{}$. For dominant weights $\lambda, \mu \in \Lambda^+ \subset \Lambda$ we have $\widetilde{\lambda}\widetilde{\mu} = \widetilde{\lambda+\mu}$, thus we have a homomorphism $\Lambda \to B_{aff}$ sending $\lambda \in \Lambda^+$ to $\widetilde{\lambda}$, we denote this homomorphism by $\lambda \mapsto \theta_\lambda$.

For a set S of objects in a triangulated category $\mathcal{C}$ we let $\langle S\rangle$ denote the full subcategory in $\mathcal{C}$ generated by S under extensions and direct summands.

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2. Noncommutative Springer resolution

In this section we summarize the results of [**5**], [**7**] (see also [**3**]).

We will use the language of DG-schemes, see [**7**] for the summary of necessary elementary facts. We only use DG-schemes explicitly presented as fiber products of ordinary schemes, so we do not require the subtler aspects of the theory discussed in the current literature on the subject.

The concept of a *geometric action* of a group on a scheme X over a scheme Y (where a map $X \to Y$ is fixed) is introduced in [**5**], [**7**]. We do not recall the definition in detail, but we mention that a geometric action induces a usual action on the derived category $DGCoh(X \overset{\mathrm{L}}{\times}_Y S)$ for any scheme S mapping to Y. Here $X \overset{\mathrm{L}}{\times}_Y S$ is the derived fiber product and $DGCoh$ denotes the triangulated category of sheaves of coherent DG-modules over the structure sheaf. In the case when higher Tor sheaves $Tor_i^{\mathcal{O}(Y)}(\mathcal{O}(X), \mathcal{O}(S))$, $i > 0$, vanish, this reduces to the usual fiber product $X \times_Y S$ and we have $DGCoh(X \overset{\mathrm{L}}{\times}_Y S) = D^b(Coh(X \times_Y S))$. For varying S, the actions are compatible with pull-back and push-forward functors.

Recall that $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is the Grothendieck-Springer map. In [**7**] a geometric action of B_{aff} on $\tilde{\mathfrak{g}}$ over $\mathfrak{g}$ is constructed.

For a quasi-projective scheme S of finite type over $\mathbb{C}$ with a fixed map to $\mathfrak{g}$ set $\tilde{\mathfrak{g}}_S = \tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} S$.

We let $\aleph$ denote the geometric action and $\aleph_S$ the corresponding action of B_{aff} on $DGCoh(\tilde{\mathfrak{g}}_S)$. The action $\aleph_S$ can be described as follows.

For $\lambda \in \Lambda$ let $\mathcal{O}_{\mathcal{B}}(\lambda)$ be the corresponding line bundle on the flag variety, and $\mathcal{O}_{\tilde{\mathfrak{g}}_S}$ be its pull-back to $\tilde{\mathfrak{g}}_S$.

For $\alpha \in \Sigma$ let $\mathcal{P}_\alpha$ be the corresponding partial flag variety thought of as the variety of parabolic subalgebras in $\mathfrak{g}$ belonging to a fixed conjugacy class. Let $\tilde{\mathfrak{g}}_\alpha = \{(x, \mathfrak{p}) \mid \mathfrak{p} \in \mathcal{P}_\alpha, x \in \mathfrak{p}\}$, and let Γ^α denote the component of $\tilde{\mathfrak{g}} \times_{\tilde{\mathfrak{g}}_\alpha} \tilde{\mathfrak{g}}$ different from the diagonal. Let $\Gamma^\alpha_S = \Gamma^\alpha \overset{\mathrm{L}}{\times}_{\mathfrak{g}} S$. Let pr_1, $pr_2 : \Gamma^\alpha_S \to \tilde{\mathfrak{g}}_S$ be the projections.

Then we have:

$$\aleph_S(\widetilde{s_\alpha}) : \mathcal{F} \to pr^\alpha_{2*} pr_1^{\alpha *}(\mathcal{F}), \qquad \alpha \in \Sigma$$

$$\aleph_S(\theta_\lambda) : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_{\tilde{\mathfrak{g}}_S}} \mathcal{O}_{\tilde{\mathfrak{g}}_S}(\lambda).$$

We say[1] that a t-structure τ on $DGCoh(\tilde{\mathfrak{g}}_S)$ is *braid positive* if $\aleph_S(\widetilde{s_\alpha})$, $\alpha \in \Sigma_{aff}$ is right exact i.e. it sends $DGCoh(\tilde{\mathfrak{g}}_S)^{\leq 0}_\tau$ to itself. Notice that this definition involves the action of $\widetilde{s_\alpha}$ for all $\alpha \in \Sigma_{aff}$; in particular, for $\alpha \notin \Sigma$ this action is not given by an explicit correspondence (though it can be expressed as a composition of correspondences used defining the action of $\widetilde{s_\alpha}^{\pm 1}$, $\alpha \in \Sigma$ and θ_λ).

We will say that such a t-structure is *normalized* if the direct image functor $R\pi_{S*} : DGCoh(\tilde{\mathfrak{g}}_S) \to D^b(Coh(S))$ is t-exact where the target category is equipped with the tautological t-structure.

The following was established in [**5**].

[1]This terminology differs slightly from that of [**5**] – a normalized braid positive t-structure was called an exotic t-structure in *loc. cit.*

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THEOREM 2. *a) For any S a normalized braid positive t-structure exists and it is unique. It satisfies:*

$\mathcal{F} \in D^{\geq 0}$ iff $pr_{S*}(b^{-1}(F)) \in D^{\geq 0}(Coh(S))$ for all $b \in B^+_{aff}$

$\mathcal{F} \in D^{\leq 0}$ iff $pr_{S*}(b(F)) \in D^{\leq 0}(Coh(S))$ for all $b \in B^+_{aff}$.

b) There exists a finite locally free $\mathcal{O}(\mathfrak{g})$ algebra A such that for any S as above there is an equivalence $D^b(A_S\text{–mod}^{fg}) \cong DGCoh(\tilde{\mathfrak{g}}_S)$ sending the tautological t-structure on the LHS to the normalized braid positive t-structure on the RHS. Here $A_S = A \otimes_{\mathcal{O}(\mathfrak{g})} \mathcal{O}(S)$ and $A_S\text{–mod}^{fg}$ is the category of finitely generated A_S-modules.

REMARK 1. It is easy to see that the properties stated in part (b) of the Theorem characterize the algebra A appearing there uniquely up to a Morita equivalence. (In fact, the part of the statement pertaining to the absolute case $S = \mathfrak{g}$ is sufficient to characterize A). For notational convenience we fix a representative A of the Morita equivalence class.

REMARK 2. The Theorem was stated in [**5**] under the additional assumption of Tor vanishing, when $\tilde{\mathfrak{g}}_S$ can be considered as an ordinary scheme rather than a DG-scheme, and only for affine S. However, the proof carries over to the case of arbitrary base change involving DG-schemes, given the foundational material in [**16**, §1], [**7**].

REMARK 3. The characterization of a normalized braid positive t-structure involves only the action of Coxeter generators $\widetilde{s}_\alpha$ which generate the semi-group B^+_{aff} if G is adjoint but not in general. However, elements of B^+_{aff} act by right exact functors for any G, see [**5**, Remark 1.5.2]. In particular, the subgroup Ω of length zero elements in W_{aff} acts by t-exact automorphisms, i.e. it acts by automorphisms of the corresponding abelian heart. Notice that Ω acts on $\widehat{\mathfrak{g}}$ by outer automorphisms coming from automorphisms of the affine Dynkin diagram. Thus if an oper $\mathfrak{O}$ is Ω-invariant we get an action of Ω on $U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}_{\mathfrak{O}}$. It is natural to conjecture that for such an oper the equivalence of Theorem 1 is compatible with the action of Ω.

2.1. Base change to a point and canonical bases. We now turn to the particular case when $S = \{e\}$ is a point. We assume for simplicity that $e \in \mathcal{N}$.

Then we get a finite dimensional algebra A_e together with the equivalence

$$D^b(A_e\text{–mod}^{fg}) \cong DGCoh(\tilde{\mathfrak{g}}_e). \tag{1}$$

The reduced variety of the DG-scheme $\tilde{\mathfrak{g}}_e$ is the *Springer fiber* $\mathcal{B}_e = \pi^{-1}(e)$. It follows that the Grothendieck group $K^0(DGCoh(\tilde{\mathfrak{g}}_e))$ is isomorphic to $K^0(Coh(\mathcal{B}_e)) =: K_0(\mathcal{B}_e)$. The equivalence (1) induces an isomorphism

$$K^0(A_e\text{–mod}^{fg}) \cong K^0(DGCoh(\tilde{\mathfrak{g}}_e)) = K^0(Coh(\mathcal{B}_e)).$$

Since A_e is a finite dimensional algebra over k, the group $K^0(A_e\text{–mod}^{fg})$ is a free abelian group with a basis formed by the classes of irreducible A_e-modules.

The following is a restatement of the main result of [**5**].

THEOREM 3. *The basis in $K_0(\mathcal{B}_e)$ corresponding to the basis of irreducible A_e-modules under the above isomorphisms is the canonical basis, i.e. it is characterized (uniquely up to signs) by the axioms of* [**14**].

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2.2. The equivariant version. Let H be a reductive group with a homomorphism $H \to G$, and assume that H acts on S so that the map $S \to \mathfrak{g}$ is H-equivariant. Then we get (see [**5**, 1.6.6])

$$DGCoh^H(\tilde{\mathfrak{g}}_S) \cong D^b(A_S\text{–mod}^H_{coh}), \tag{2}$$

where A_S–mod^H_{coh} denotes the category H-equivariant finitely generated A_S-modules.

Below we will apply it in the case $S = \tilde{\check{\mathcal{N}}}$, $H = G$.

3. Perverse sheaves on the affine flag variety

3.1. Affine flag variety and categories of constructible sheaves. Along with the Lie algebra $\check{\mathfrak{g}}((t))$ we consider the group ind-scheme $\check{G}((t))$ and its group subschemes $I^0 \subset I \subset \mathbf{G_O}$, where I is the Iwahori subgroup and I^0 is its pro-unipotent radical and $\mathbf{G_O} = \check{G}[[t]]$ is the subgroup of regular loops into $\check{G}$. Let also $I_- \subset \mathbf{G_O}$ be an opposite Iwahori and I^0_- be its pro-unipotent radical. Let $\mathcal{F}\ell$ be the affine flag variety for the group $\check{G}$; thus $\mathcal{F}\ell = \check{G}((t))/I$ is an ind-projective ind-scheme.

We will consider the following full subcategories in the derived category $D(\mathcal{F}\ell)$ of constructible sheaves on $\mathcal{F}\ell$: the category $D_{I^0}(\mathcal{F}\ell)$ of complexes equivariant with respect to I^0 and D_{IW} of complexes equivariant with respect to a non-degenerate character of I^0_-, see [**1**] for details. The functors of forgetting the equivariance $D_{I^0}(\mathcal{F}\ell) \to D(\mathcal{F}\ell)$, $D_{IW} \to D(\mathcal{F}\ell)$ are full embeddings since the group schemes I^0, I^0_- are pro-unipotent.

Let $Perv_{I^0}(\mathcal{F}\ell) \subset D_{I^0}(\mathcal{F}\ell)$, $Perv_{IW}(\mathcal{F}\ell) \subset D_{IW}(\mathcal{F}\ell)$ be the full subcategories of perverse sheaves. It is known that there are natural equivalence $D^b(Perv_{I^0}(\mathcal{F}\ell)) \cong D_{I^0}(\mathcal{F}\ell)$, $D^b(Perv_{IW}(\mathcal{F}\ell)) \cong D_{IW}(\mathcal{F}\ell)$ (see e.g. [**1**]).

We will also need the Iwahori equivariant derived category $D_I(\mathcal{F}\ell)$ (which, in contrast with the categories $D_{I^0}(\mathcal{F}\ell)$, D_{IW} is not equivalent to the derived category of the abelian subcategory of perverse sheaves $Perv_I(\mathcal{F}\ell)$). The category $D_I(\mathcal{F}\ell)$ carries a monoidal structure provided by convolution which will be denoted by $\star$. This monoidal category acts on $D(\mathcal{F}\ell)$ by convolution on the right, which will also be denoted by $\star$; the action preserves the subcategories D_{I^0}, D_{IW}.

The orbits of I on $\mathcal{F}\ell$ are indexed by W_{aff}; for $w \in W_{aff}$ let $\mathcal{F}\ell_w$ denote the corresponding orbit and $j_w : \mathcal{F}\ell_w \to \mathcal{F}\ell$ be the embedding. We abbreviate $j_{w*} = j_{w*}(\underline{\mathbb{C}}[\dim \mathcal{F}\ell_w])$, $j_{w!} = j_{w!}(\underline{\mathbb{C}}[\dim \mathcal{F}\ell_w])$; thus j_{w*}, $j_{w!} \in Perv_I(\mathcal{F}\ell) \subset D_I(\mathcal{F}\ell)$.

We have $j_{w_1*} \star j_{w_2*} \cong j_{w_1w_2*}$, $j_{w_1!} \star j_{w_2!} \cong j_{w_1w_2!}$, $j_{w_1w_2*} \star j_{w_2^{-1}!} \cong j_{w_1*}$ provided that $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.

For $\lambda \in \Lambda$ the corresponding *Wakimoto sheaf* $J_\lambda \in Perv_I(\mathcal{F}\ell) \subset D_I(\mathcal{F}\ell)$ is introduced in [**1**, 3.2]. It can be characterized by $J_\lambda \star J_\mu \cong J_{\lambda+\mu}$ for λ, $\mu \in \Lambda$ and $J_\lambda \cong j_{\lambda*}$ for dominant λ. Notice that $J_\lambda \cong j_{\lambda!}$ when λ is antidominant.

The orbits of $\mathbf{G_O}$ on $\mathcal{F}\ell$ are indexed by Λ, we let $\mathcal{F}\ell^\lambda$ denote the orbit corresponding to $\lambda \in \Lambda$ and let $i_\lambda : \mathcal{F}\ell^\lambda \to \mathcal{F}\ell$ be the embedding. There exists a unique irreducible Iwahori-Whittaker perverse sheaf on $\mathcal{F}\ell^\lambda$, we let Δ_λ (respectively, ∇_λ) be its ! (respectively, $*$) extension to $\mathcal{F}\ell$. We have $\Delta_\lambda = \Delta_0 \star j_{w!}$, $\nabla_\lambda = \Delta_0 \star j_{w*}$ if $w \in W \cdot \lambda$.

We set also $J^{IW}_\lambda = \Delta_0 \star J_\lambda$. The functor $\mathcal{F} \mapsto \Delta_0 \star \mathcal{F}$ is t-exact [**1**, Proposition 2], so we have J^{IW}_λ, Δ_λ, $\nabla_\lambda \in Perv_{IW}$.

Recall the *central sheaves* Z_λ, $\lambda \in \Lambda^+$ of [**12**].

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3.2. The equivalence of [1] and its relation to the t-structures. The main result of [**1**] is a construction of an equivalence of triangulated categories

$$\Phi_{IW} : D^b(Coh^G(\tilde{\mathcal{N}})) \cong D_{IW}(\mathcal{F}\ell). \tag{3}$$

We recall some properties of the Φ_{IW} that will be used below.

$$\Phi_{IW}(\mathcal{O}_{\tilde{\mathcal{N}}}) = \Delta_0, \tag{4}$$

$$\Phi_{IW}(\mathcal{F} \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \cong \Phi_{IW}(\mathcal{F}) \star J_\lambda, \tag{5}$$

$$\Phi_{IW}(\mathcal{F} \otimes V_\lambda) \cong \Phi_{IW}(\mathcal{F}) \star Z_\lambda, \tag{6}$$

where V_λ denotes the irreducible G-module with highest weight λ.

The following technical statement relating the natural t-structures in the two sides of (3) will play a key role in the proof of the main result.

Proposition 1. *For $\mathcal{F} \in D_{IW}(\mathcal{F}\ell)$ the following are equivalent:*
i) For all $\lambda \in \Lambda$, $\mathcal{F} \star J_\lambda \in D^{\leq 0}(Perv_{IW}(\mathcal{F}\ell))$.
ii) $\mathcal{F} \in \langle J^{IW}_\lambda[d] \mid d \geq 0\rangle$.
iii) $\Phi^{-1}_{IW}(\mathcal{F}) \in D^{\leq 0}(Coh^G(\tilde{\mathcal{N}}))$, where $D^{\leq 0}$ is taken with respect to the tautological t-structure on $D^b(Coh^G(\tilde{\mathcal{N}}))$.

Proof. ii) $\Rightarrow$ iii) follows from (4), (5) which imply that $\Phi^{-1}_{IW}(J^{IW}_\lambda) = \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$.

To check that iii) $\Rightarrow$ ii) notice that every equivariant coherent sheaf on a quasi-projective algebraic variety with a reductive group action is a quotient of a line bundle tensored by a representation of the group. In particular, every object in $Coh^G(\tilde{\mathcal{N}})$ is a quotient of a sheaf of the form $V \otimes \mathcal{O}(\lambda)$ for some $V \in Rep(G)$. It follows by a standard argument that every object $D^{\leq 0}(Coh^G(\tilde{\mathcal{N}})) \cap D^b(Coh^G(\tilde{\mathcal{N}}))$ is a direct summand in an object represented by a finite complex of sheaves of the form $V \otimes \mathcal{O}(\lambda)$ concentrated in non-positive degrees. Thus $D^{\leq 0}(Coh^G(\tilde{\mathcal{N}})) \cap D^b(Coh^G(\tilde{\mathcal{N}})) = \langle V_\nu \otimes \mathcal{O}(\lambda)[d]\rangle$, $d \geq 0$. So we see that $\Phi^{-1}_{IW}(\mathcal{F}) \in D^{\leq 0}(Coh^G(\tilde{\mathcal{N}}))$ iff $\mathcal{F} \in \langle J^{IW}_\lambda \star Z_\nu\rangle$. Since Z_ν admits a filtration with associated graded being a sum of Wakimoto sheaves [**1**, 3.6] and $J^{IW}_\lambda \star J_\mu \cong J^{IW}_{\lambda+\mu}$, we get that $\langle J^{IW}_\lambda \star Z_\nu\rangle = \langle J^{IW}_\lambda\rangle$, which yields the implication.

The implication ii) $\Rightarrow$ i) is clear from $J^{IW}_\lambda \in Perv_{IW}$, $J^{IW}_\lambda \star J_\mu \cong J^{IW}_{\lambda+\mu}$.

Finally to check that i) $\Rightarrow$ ii) we need an auxiliary statement.

Lemma 1 (see [**1**, Lemma 15]). *Given $\mathcal{F} \in D_{IW}(\mathcal{F}\ell)$ there exists a finite subset $\mathfrak{S} \subset \Lambda$, such that for μ, $\lambda \in \Lambda$ we have $i^*_\mu(\mathcal{F} \star j_{\nu!}) = 0$ unless $\mu \in \mathfrak{S} + \nu$.* □

Now, to check i) $\Rightarrow$ ii) let $\mathfrak{S} \subset \Lambda$ be constructed as in the Lemma, and let $\nu \in \Lambda$ be such that both $\{\nu\}$ and $\mathfrak{S} + \nu$ are contained in the set of antidominant weights. Using the standard exact triangles connecting a constructible complex, its $*$ restriction to a closed subset and ! extension from the open complement we see that

$$\mathcal{F} \star j_{\nu!} \in \langle \Delta_\lambda[d] \mid \lambda \in \mathfrak{S} + \nu, d \in \mathbb{Z}\rangle. \tag{7}$$

Since ν is antidominant, we have $J_\nu = j_{\nu!}$; so condition i) says that $\mathcal{F} \star j_{\nu!} \in D^{\leq 0}(Perv_{IW}(\mathcal{F}\ell))$. Given (7), this is equivalent to $\mathcal{F} \star j_{\nu!} \in \langle \Delta_\lambda[d] \mid \lambda \in \mathfrak{S} + \nu, d \geq 0\rangle$. Since all λ appearing in the last expression are antidominant, we get that $\mathcal{F} \star J_\nu \in \langle J^{IW}_\lambda[d] \mid d \geq 0\rangle$ which yields ii). □

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3.3. The B_{aff} action on $D_{I^0}(\mathcal{F}\ell)$. The two sided cosets of I in $G\check{}((t))$ are indexed by W_{aff}. For each $w \in W_{aff}$ fix a representative $\mathbf{w} \in IwI$ (it is easy to check that the functors defined in the Lemma below do not depend on this choice, up to a noncanonical isomorphism). Let ${}^wI^0 = \mathbf{w}I^0\mathbf{w}^{-1}$. Let $Conv_w$ denote the quotient of $I^0 \times \mathcal{F}\ell$ by the action of $I^0 \cap^w I^0$ given by $g : (\gamma, x) \mapsto (\gamma g^{-1}, g(x))$. The action map descends to a map $conv_w : Conv_w \to \mathcal{F}\ell$.

For $\mathcal{F} \in D_{I^0}(\mathcal{F}\ell)$ the complex $\mathbf{w}_*(\mathcal{F})$ is equivariant with respect to $I^0 \cap^w I^0$, thus the complex $\underline{\mathbb{C}} \boxtimes \mathbf{w}_*\mathcal{F}$ on $I \times \mathcal{F}\ell$ descends to a canonically defined complex on $Conv_w$, let us denote it by $\mathcal{F}_w$. The following is standard.

LEMMA 2. *There exists an (obviously unique) action of B_{aff} on $D_{I^0}(\mathcal{F}\ell)$, such that for $w \in W_{aff}$,*

$$\widetilde{w} : \mathcal{F} \mapsto conv_{w*}(\mathcal{F}_w)[\ell(w)].$$

It satisfies:

$$\widetilde{w}^{-1} : \mathcal{F} \mapsto conv_{w^{-1}!}(\mathcal{F}_w)[\ell(w)].$$

3.4. A coherent description of $D_{I^0}(\mathcal{F}\ell)$. A proof of the following result will appear in [**4**], see also announcement in [**3**].

Define the *Steinberg variety* as $St = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$. Let $Av_{IW} : D_{I^0}(\mathcal{F}\ell) \to D_{IW}(\mathcal{F}\ell)$ be the *averaging* functor, i.e. the adjoint functor to the embedding of $D_{IW}(\mathcal{F}\ell)$ into the category of constructible complexes on $\mathcal{F}\ell$, restricted to $D_{I^0}(\mathcal{F}\ell)$.

THEOREM 4. *There exists an equivalence of triangulated categories*

$$\Phi : D^b(Coh^G(St)) \cong D_{I^0}(\mathcal{F}\ell),$$

satisfying the following properties.

a) Φ intertwines the B_{aff} action from section 2 *with that from* 3.3.

b) $pr_{2*} \circ \Phi^{-1} \cong \Phi_{IW}^{-1} \circ Av_{IW}$.

We refer the reader to [**3**] and [**9**] for a discussion of some other properties of this equivalence.

3.5. The "new t-structure". We are now in the position to derive a partial answer to a question of [**9**].

PROPOSITION 2. *For $\mathcal{F} \in D_{I^0}(\mathcal{F}\ell)$ the following are equivalent*

i) For all $\lambda \in \Lambda$ we have $\mathcal{F} \star J_\lambda \in D^{\leq 0}(Perv_{I^0}(\mathcal{F}\ell))$.

i′) There exists λ_0 such that $\mathcal{F} \star J_\lambda \in D^{\leq 0}(Perv_{I^0}(\mathcal{F}\ell))$ for $\lambda \in \lambda_0 - \Lambda^+$.

ii) $\Phi^{-1}(\mathcal{F}) \in D^{\leq 0}(Coh^G(St))$ where $D^{\leq 0}(Coh^G(St))$ is equipped with the braid positive normalized t-structure for $S = \tilde{\mathcal{N}}$.

PROOF. i) $\Rightarrow$ i′) is obvious. To see that i′) $\Rightarrow$ i) notice that any $\lambda \in \Lambda$ can be written as $\lambda = \lambda' + \mu$ where $\mu \in \Lambda^+$ and $\lambda' \in \lambda_0 - \Lambda^+$; then $\mathcal{F} \star J_\lambda = (\mathcal{F} \star J_{\lambda'}) \star j_{\mu*} \in D^{\leq 0}(Perv_{I^0}(\mathcal{F}\ell))$ since convolution with j_{w*} is right exact as it amounts to taking direct image under an affine morphism.

It remains to show that i) $\Leftrightarrow$ ii). Assume that i) holds for a given $\mathcal{F}$. In view of Theorem 2a) and compatibility of the B_{aff} actions, we need to check that for all $b \in B_{aff}^+$, $pr_*(\Phi^{-1}(b(\mathcal{F}))) \in D^{\leq 0}(Coh(\tilde{\mathcal{N}}))$. Since B_{aff}^+ acts on $D^b(Perv_{I^0}(\mathcal{F}\ell))$ by left exact functors, $b(\mathcal{F}) \star J_\lambda \in D^{\leq 0}(Perv_{I^0}(\mathcal{F}\ell))$ for all $b \in B_{aff}^+$, $\lambda \in \Lambda$. Thus $Av_{IW}(b(\mathcal{F})) \star J_\lambda \in D^{\leq 0}(Perv_{IW}(\mathcal{F}\ell))$ for all λ. Applying Proposition 1 to $Av_{IW}(b(\mathcal{F})) \star J_\lambda$ we see that $\Phi_{IW}^{-1}(Av_{IW}(b(\mathcal{F}))) = pr_*(b(\Phi^{-1}(\mathcal{F}))) \in D^{\leq 0}(Coh^G(\tilde{\mathcal{N}}))$, which gives ii).

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The converse statement follows from Proposition 1 and the following

PROPOSITION 3. *Suppose $\mathcal{F} \in D_{I^0}(\mathcal{F}\ell)$ is such that $Av_{IW}(b(\mathcal{F})) \in D^{\leq 0}(Perv_{IW}(\mathcal{F}\ell))$ for all $b \in B^+_{aff}$. Then $\mathcal{F} \in D^{\leq 0}(Perv_{I^0}(\mathcal{F}\ell))$.*

PROOF. Let $\mathcal{F}$ be as in the statement of the Proposition, and let n be the smallest integer such that $\mathcal{F} \in D^{\leq n}_{I^0}(\mathcal{F}\ell)$. We need to show that $n \leq 0$.

Let $\mathcal{F}_n$ be the n-th perverse cohomology sheaf and L be an irreducible quotient of $\mathcal{F}_n$. It suffices to show that

$$ {}^{perv}H^0(Av_{IW}(b(L))) \neq 0 \tag{8}$$

for some $b \in B^+_{aff}$: then using right exactness of the action of $b \in B^+_{aff}$ we see that the composed arrow $b(\mathcal{F}) \to b(\mathcal{F}_n[-n]) \to b(L[-n])$ induces a surjection on the n-th perverse cohomology sheaf, hence (8) implies that n-th perverse cohomology of $Av_{IW}(b(\mathcal{F}))$ does not vanish, thus $n \leq 0$.

We now check (8). First we claim that there exists $b \in B^+_{aff}$ such that $Av_{IW}(b(L)) \neq 0$. For this we need the following variation of Lemma 1.

LEMMA 3. *Given $\mathcal{F} \in D_{I^0}(\mathcal{F}\ell)$ there exists a finite subset $\mathfrak{S} \subset W_{aff}$, such that for w_1, $w_2 \in W_{aff}$ we have $j^!_{w_2}(\tilde{w}_1(\mathcal{F})) = 0$ unless $w_2 \in w_1 \cdot \mathfrak{S}$.*

PROOF. is similar to that of Lemma 1 (see [**1**, Lemma 15]). □

Now take $b = \tilde{\lambda}$ for a dominant weight λ. We can assume that if $w = \mu \cdot w_f \in \mathfrak{S}$ where $\mathfrak{S}$ is as in the Lemma with $\mathcal{F} = L$, $w_f \in W$, $\mu \in \Lambda$, then $\lambda + \mu$ is strictly dominant. Then each left coset of W in W_{aff} contains at most one element such that the ! restriction of $\tilde{\lambda}(L)$ to the corresponding I orbit is non-zero (no cancelations in the spectral sequence containing exactly one non-zero entry). If such an element exists then the corresponding costalk of $Av_{IW}(\tilde{\lambda}(L))$ does not vanish; thus $Av_{IW}(\tilde{\lambda}(L)) \neq 0$ for such λ.

Choose now $b \in B^+_{aff}$ such that $Av_{IW}(b(L)) \neq 0$ and b is an element of minimal possible length satisfying this property. Notice that L is I-equivariant, and for an I-equivariant complex L we have $\tilde{w}(L) \cong j_{w*} \star L$ where $\star$ denotes convolution of I-equivariant complexes on $\mathcal{F}\ell$. In particular, when $w = s_\alpha$ is a simple reflection we have $\tilde{s_\alpha}(L) = j_{s_\alpha *} \star L$, $\tilde{s_\alpha}^{-1}(L) = j_{s_\alpha !} \star L$. The perverse sheaves $j_{s_\alpha !}$, $j_{s_\alpha *}$ are concentrated on the closure of a one dimensional I-orbit; this closure can be identified with $\mathbb{P}^1$, and we denote it by $\mathbb{P}^1_\alpha$. We have an exact sequence of perverse sheaves on $\mathbb{P}^1_\alpha$:

$$0 \to \delta_e \to j_{s_\alpha !} \to j_{s_\alpha *} \to \delta_e \to 0,$$

where δ_e denotes the sky-scraper at the zero-dimensional I-orbit $\{e\} \subset \mathbb{P}^1_\alpha$. This exact sequence shows that $\tilde{s_\alpha}(L)$ and $\tilde{s_\alpha}^{-1}(L)$ are isomorphic in the quotient modulo the thick subcategory generated by L. Let $b = \tilde{s}_{\alpha_1} \cdots \tilde{s}_{\alpha_n}$ be a minimal decomposition of b. Then our assumptions on b imply that

$$Av_{IW}(b(L)) \cong Av_{IW}(b'(L)),$$

where $b' = \tilde{s}^{-1}_{\alpha_1} \cdots \tilde{s}^{-1}_{\alpha_n}$. Since the action of b is right exact, the action of b' is left exact and Av_{IW} is exact, we see that $Av_{IW}(b(L))$ is a perverse sheaf; thus the assumption on b implies (8). □

REMARK 4. The idea of the proof is partly borrowed from [**5**, 2.2].

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Corollary 1. *There exists a t-structure τ_{new} on $D_{I^0}(\mathcal{F}\ell)$ given by: $\mathcal{F} \in D^{\leq 0,\mathrm{new}}$ iff $\mathcal{F} \star J_\lambda \in D^{\leq 0}(Perv_{I^0}(\mathcal{F}\ell))$. The composed equivalence*

$$D_{I^0}(\mathcal{F}\ell) \underset{\Phi}{\widetilde{\longleftarrow}} D^b(Coh^G(St)) \underset{(2)}{\widetilde{\longrightarrow}} D^b(A_{\tilde{\mathcal{N}}}\text{–mod}^G_{coh})$$

sends τ_{new} to the tautological t-structure on $D^b(A_{\tilde{\mathcal{N}}}\text{–mod}^G_{coh})$.

Remark 5. Corollary provides a positive partial answer to Question 2.1.3 of [**9**]. In more detail, in *loc. cit.* the so called new t-structure is defined on a certain Ind-completion of the bounded derived category of finitely generated D-modules on $\mathcal{F}\ell$. Then the question is posed whether this t-structure induces one on the original (not completed) bounded derived category (in a footnote the authors say they expect a negative answer). The above Corollary gives a positive answer to a weaker question: it shows that the new t-structure of [**9**] induces one on the bounded derived category of I^0-equivariant finitely generated D-modules on $\mathcal{F}\ell$.

4. Results of Frenkel-Gaitsgory and a proof of Theorem 1

4.1. The functor to modules. Recall the notion of a nilpotent oper on a punctured formal disc [**11**] (see also [**2**] for a general introduction to the notion of an oper); let Op_{nilp} be the infinite dimensional scheme parameterizing such opers. By definition $O \in Op_{\mathrm{nilp}}$ is a collection of data $O = (\mathcal{E}, F, \nabla)$ where $\mathcal{E}$ is a G-bundle on the formal disc $\mathfrak{D} = Spec(\mathbb{C}[[t]])$, ∇ is a connection on $\mathcal{E}$, having a first order pole at the origin $x_0 \in \mathfrak{D}$, $\nabla : ad(\mathcal{E}) \to t^{-1}ad(\mathcal{E})\Omega^1_{\mathfrak{D}}$; and F is a B-structure on the bundle $ad(\mathcal{E})$, these should satisfy a certain compatibility condition.

The compatibility implies in particular that the residue of the connection is nilpotent and preserves the B-structure on the fiber at the closed point x_0; thus we get a canonical map $Op_{\mathrm{nilp}} \to \tilde{\mathcal{N}}/G$. We will say that a point $(e, \mathfrak{b}) \in \tilde{\mathcal{N}}$ is compatible with a given nilpotent oper if it lies in the corresponding G-orbit.

The space of all opers maps isomorphically to the spectrum of center of the category $U_{crit}\widehat{\mathfrak{g}}$–mod by [**8**].

The following result is a direct consequence of [**9**] compared to Proposition 2.

We identify $D_{I^0}(\mathcal{F}\ell)$ with the category of I^0 equivariant critically twisted D-modules on $\mathcal{F}\ell$, this is possible by Riemann-Hilbert correspondence, since the critical twisting is integral. Then we get the derived functor of global sections from $D_{I^0}(\mathcal{F}\ell)$ to the derived category of $U_{crit}\widehat{\mathfrak{g}}$-modules; in fact it lands in the derived category of I^0 monodromic modules [**9**].

Recall that $\mathcal{B}_e = \{e\} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \tilde{\mathfrak{g}}$.

Theorem 5. *a) Fix $O \in Op_{\mathrm{nilp}}$ and $e \in \mathcal{N}$ so that $Res(O)$ is in the conjugacy class of e. Then there exists an equivalence*

$$\Phi_O : DGCoh(\mathcal{B}_e) \widetilde{\longrightarrow} D^b(U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}_O).$$

b) Fix $\tilde{e} = (e, \mathfrak{b}) \in \tilde{\mathcal{N}}$ compatible with O. For $\mathcal{F} \in DGCoh(\mathcal{B}_e)$ we have: $\Phi_O(\mathcal{F}) \in D^{>0}(U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}_O)$ iff $\mathrm{Hom}_{D^bCoh(St)}(\mathcal{G}, i_{\tilde{e}*}\mathcal{F}) = 0$ for any $\mathcal{G} \in D^b(Coh^G(St))$ which belongs to $D^{\leq 0}$ with respect to the braid positive normalized t-structure with $S = \tilde{\mathcal{N}}$. Here $i_{\tilde{e}}$ is the composed map $\tilde{\mathfrak{g}} \overset{L}{\times}_{\mathfrak{g}} \{e\} \widetilde{\longrightarrow} St \overset{L}{\times}_{\tilde{\mathcal{N}}} \{\tilde{e}\} \to St = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$.

Proof. Part (a) is [**9**, Corollary 0.6].

To check (b) we need to recall the idea of the proof of [**9**, Corollary 0.6]. That result is obtained by combining our equivalence of Theorem 4 with the equivalence

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of *loc. cit.*, Main Theorem 2 between a certain Ind-completion of $D^b(U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0})$ and an (appropriately defined) base change of the category $D_{I^0}(\mathcal{F}\ell)$ with respect to the morphism $Op_{\mathrm{nilp}} \to \tilde{\mathcal{N}}/G$; here the construction of [**1**] is used to endow $D_{I^0}(\mathcal{F}\ell)$ with the structure of a category over $\tilde{\mathcal{N}}/G$ (see [**9**] and references therein for a definition of a category over a stack and the notion of base change in this context). Thus, using the usual notation for base change (and omitting Ind-completion from notation), [**9**, Main Theorem 2] asserts that:

$$D^b(U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0}) \cong Op_{\mathrm{nilp}} \times_{\tilde{\mathcal{N}}/G} D_{I^0}(\mathcal{F}\ell). \tag{9}$$

Furthermore, the category $D^b(U_{crit}\widehat{\mathfrak{g}}\text{–mod}_O^{I^0})$ is obtained from $D^b(U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0})$ by base change with respect to the morphism of point embedding $\{O\} \to Op_{\mathrm{nilp}}$. Thus we get

$$D^b(U_{crit}\widehat{\mathfrak{g}}\text{–mod}_O^{I^0}) \cong \{O\} \times_{Op_{\mathrm{nilp}}} (Op_{\mathrm{nilp}} \times_{\tilde{\mathcal{N}}/G} D_{I^0}(\mathcal{F}\ell)) \cong \{O\} \times_{\tilde{\mathcal{N}}/G} D_{I^0}(\mathcal{F}\ell).$$

Substituting the equivalence of Theorem 4: $D_{I^0}(\mathcal{F}\ell) \cong D^b(Coh^G(St))$ we can rewrite the latter category as

$$\{O\} \times_{\tilde{\mathcal{N}}/G} D^b(Coh(St/G)) \cong DGCoh(\{O\} \overset{\mathrm{L}}{\times}_{\tilde{\mathcal{N}}/G} St/G)$$

where the equivalence comes from basic properties of base change for categories. Finally,

$$(St/G) \overset{\mathrm{L}}{\times}_{\tilde{\mathcal{N}}/G} \{O\} \cong (\tilde{\mathfrak{g}}/G) \times_{\mathfrak{g}/G} (\tilde{\mathcal{N}}/G) \overset{\mathrm{L}}{\times}_{\tilde{\mathcal{N}}/G} \{O\} \cong \tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \{O\} \cong \tilde{\mathfrak{g}} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \{e\},$$

which gives the desired equivalence. Notice that the tautological functor $D^b(U_{crit}\widehat{\mathfrak{g}}\text{–mod}_O^{I^0}) \to D^b(U_{crit}\widehat{\mathfrak{g}}\text{–mod}^{I^0})$ corresponds under the above equivalences to the push-forward functor $i_{O*} : \{O\} \times_{\tilde{\mathcal{N}}/G} D_{I^0}(\mathcal{F}\ell) \to Op_{\mathrm{nilp}} \times_{\tilde{\mathcal{N}}/G} D_{I^0}(\mathcal{F}\ell)$ which comes from the point embedding $i_O : \{O\} \to Op_{\mathrm{nilp}}$ via functoriality of the base change construction.

We are now ready to deduce statement (b) from the exactness statement in [**9**, Main Theorem 2]. The latter yields the following description of

$\Phi_O^{-1}(D^{>0}(U_{crit}\widehat{\mathfrak{g}}\text{–mod}_O^{I^0}))$ (cf. definition of the t-structure in *loc. cit.*, 3.6.1):

for $\mathcal{F} \in DGCoh(\{e\} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \tilde{\mathfrak{g}})$ we have $\Phi_O(\mathcal{F}) \in D^{>0}(U_{crit}\widehat{\mathfrak{g}}\text{–mod}_O^{I^0})$ iff the following holds. For any $\mathcal{G} \in D_{I^0}(\mathcal{F}\ell)$ such that $J_\lambda \star \mathcal{G} \in D^{\leq 0}(Perv_{I^0}(\mathcal{F}\ell))$ for all λ we have

$$\mathrm{Hom}_{Op_{\mathrm{nilp}} \times_{\tilde{\mathcal{N}}/G} D_{I^0}(\mathcal{F}\ell)}(pr_2^*(\mathcal{G}), i_{O*}\mathcal{F}') = 0, \tag{10}$$

where $\mathcal{F}'$ is the image of $\mathcal{F}$ under the equivalence $DGCoh(\{e\} \overset{\mathrm{L}}{\times}_{\mathfrak{g}} \tilde{\mathfrak{g}}) \cong \{0\} \times_{\tilde{\mathcal{N}}/G} D_{l^0}(\mathcal{F}\ell)$, and pr_2^* denotes the natural pull-back functor $D_{I^0}(\mathcal{F}\ell) \to Op_{\mathrm{nilp}} \times_{\tilde{\mathcal{N}}/G} D_{I^0}(\mathcal{F}\ell)$.

Since the composed morphism $pt \to \tilde{\mathcal{N}}/G$ is isomorphic to the composition $\{\tilde{e}\} \to \tilde{\mathcal{N}} \to \tilde{\mathcal{N}}/G$, we can use projection formula to simplify the last condition to:

$$\mathrm{Hom}_{DGCoh(\{\tilde{e}\} \overset{\mathrm{L}}{\times}_{\tilde{\mathcal{N}}} St)}(i_{\tilde{e}}^*(\Phi^{-1}(\mathcal{G})), \mathcal{F}) = 0,$$

where $i_{\tilde{e}}$ is the map $\{\tilde{e}\} \overset{\mathrm{L}}{\times}_{\tilde{\mathcal{N}}} St \to St$. In view of Proposition 2 we get the result. □

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4.2. The proof of Theorem 1. In view of Theorem 5 it suffices to check that the image of $D^{>0}(A_e\text{–mod}^{fg})$ under equivalence (1) consists of such objects $\mathcal{F} \in DGCoh(\tilde{\mathfrak{g}}_e)$ that $\mathrm{Hom}(\mathcal{G}, i_{\tilde{e}*}\mathcal{F}) = 0$ when $\mathcal{G} \in D^b(Coh^G(St))$ lies in $D^{\leq 0}$ with respect to the normalized braid positive t-structure with $S = \tilde{\mathcal{N}}$. This is immediate from Theorem 2. □

Remark 6. The equivalences of [**9**] are based on existence of a certain infinite dimensional vector bundle on the space of Miura opers carrying an action of $\widehat{\mathfrak{g}}$ at the critical level. According to a conjecture of [**9**] the fibers of this bundle are baby Wakimoto modules for $\widehat{\mathfrak{g}}$ at the critical level.

Likewise, the equivalence of [**6**], [**5**] are based on the existence of certain vector bundles on the space $\tilde{\mathfrak{g}}$ and its subspaces. In particular, in [**6**] a certain vector bundle on the formal neighborhood of a Springer fiber in $\tilde{\mathfrak{g}}$ over a field of positive characteristic k is constructed. It carries an action of the Lie algebra $\mathfrak{g}_k$ and its fibers are baby Verma modules for $\mathfrak{g}_k$.

Theorem 1 shows that the two vector bundles are related. In particular, consider their pull-back to a fixed Springer fiber (defined as either derived or an ordinary scheme) where the Springer fiber is embedded in the space of Miura opers by fixing a point in Op_{nilp}. Then the bundle of baby Verma modules in positive characteristic is a sum of indecomposable summands; for almost all values of $p = char(k)$ each such summand can be lifted to characteristic zero, and the bundle of $U_{crit}\widehat{\mathfrak{g}}$ modules is the sum of the resulting indecomposable bundles (generally with infinite multiplicities).

It would be interesting to find a more direct explanation of this phenomenon.

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