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Introduction

In these lectures we would like to touch on the mysterious role that groups, especially Lie groups, play in revealing the laws of nature. We will try to illustrate the hidden role of certain groups by focusing on a single and familiar example. In classical mechanics the example manifests itself as Kepler motion, the motion of a planet under the attraction of the sun according to Kepler's laws. Newton realized that Kepler's second law, equal areas are swept out in equal times, has to do with the fact that the force is directed radially to the sun. Kepler's second law is really the assertion of the conservation of angular momentum, and this reflects the rotational symmetry of the system about the origin of the force. Kepler's second law is true for any classical mechanical system exhibiting this rotational symmetry. In today's language we would say that the group $O(3)$ (the orthogonal group in three dimensions) is responsible for Kepler's second law. But Newton also realized that Kepler's first and third laws are special to the inverse square law of attraction. By the end of the nineteenth century it was realized (by Runge and Lenz) that Kepler's first and third laws have to do with the group $O(4)$—that the inverse square law of attraction has an $O(4)$ symmetry, where $O(4)$ denotes the orthogonal group in four dimensions. But this four-dimensional orthogonal group does not act on ordinary three-dimensional space. Rather, it acts on a portion of the six-dimensional phase space of the planet, the portion that describes planetary (as opposed to hyperbolic) motion. (In fact, as we shall see, the story is a bit more complicated, in that one must "complete" the phase space by including collision orbits so that the planet can pass through the sun.) In this century it has been realized that even larger groups are involved in Kepler motion. For example we will see that the fifteen-dimensional group $O(2, 4)$ of all orthogonal transformations of six space preserving a quadratic form of signature $(2, 4)$ plays a key role and even the symplectic groups in eight- and twelve-dimensional space get involved. In quantum mechanics our example manifests itself as the "hydrogen atom." Indeed, in 1925 (before Schrodinger published his famous equation!) Pauli derived the spectrum of the hydrogen atom by following the procedures of Lenz and Runge, but where the "Poisson brackets" of classical mechanics are replaced by the "commutator brackets" of quantum mechanics. Once again, as was realized by Fock, it is the group O(4) which is behind the very special character of the spectrum of the hydrogen atom. We have tried to write the first part of these lectures with the general mathematical reader in mind. So the first six sections and the beginning of section seven should not make heavy technical demands. We have also added two appendices to round out the picture for the general reader.

Let us give a short summary of the contents for the specialist. The thrust of the first six sections is to show that the Kepler problem and the hydrogen atom exhibit o(4) symmetry and that the form of this o(4) symmetry determines the inverse square law in classical mechanics and the spectrum of the hydrogen atom in quantum mechanics. All this is in the spirit of the classical treatment of Runge, Lenz, Pauli, Fock, and Moser. The space of regularized elliptical motions of the Kepler problem is symplectically equivalent to T^+S^3, the space of nonzero covectors in T^*S^3 as was realized by Souriau who called T^+S^3 the *Kepler manifold*. This manifold plays the central role in this monograph. It is connected with the Howe pairs, ([**H89**]), $O(2,4) \times Sl(2,\mathbf{R}) \subset Sp(12,\mathbf{R})$ and $U(2,2) \times U(1) \subset sp(8,\mathbf{R})$. According to the general theory of the classical mechanics of such pairs, [**KKS78**], it can be realized as a (component of a) coadjoint orbit of the first factor or a reduction of the second factor. As a coadjoint orbit of SO(2, 4) or SU(2, 2) it is the minimal nilpotent coadjoint orbit of these locally isomorphic groups; hence the problem of its quantization can be regarded as an instance of the interesting question of representations associated with such orbits. As a Marsden-Weinstein reduction at 0 of $Sl(2,\mathbf{R})$, the principle of reduction in stages shows that T^+S^3 can be regarded as the space of forward null geodesics on the conformal completion, M, of Minkowski space. In §§13–21 we study the various cosmological models in this same conformal class (and having varying isometry groups) from the viewpoint of projective geometry. On the other hand, the Kepler Hamiltonian can be derived by reduction from a geodesic flow in five dimensions, applying [**I81**] a general formula for the phase space of a classical particle moving in the presence of a Yang-Mills field; see [**S77a**] and [**We78**]. The principle of quantization of constraints [**D64**] can then be used to compute the hydrogen spectrum [**M89**]. Thus we have an illustration of the principle put forward in [**KKS78**] that enlarging the phase space can simplify the equations of motion in the classical setting and aid in the quantization problem in the quantum setting. The commutativity of quantization and reduction was worked out in the Kahler setting in [**GS82c**]; for a recent application of this method in an infinite-dimensional situation see [**APW89**]. A short summary of the homological quantization of constraints following [**KS87**] and a list of recent applications of this method to many interesting finite-dimensional settings, [**DET89**] and [**DEGST90**], is given in §12. Finally, in §22 we outline Kostant's theory, in which a unitary representation is associated to the minimal nilpotent orbit of SO(4, 4), and in which electromagnetism and gravitation are unified in a Kaluza-Klein type theory in six dimensions.

Much of the work illustrated here represents joint research with Kostant. We would like to thank Drs. Duval, Elhadad, and Tuynman for supplying us with the details of the computations in **[DET89]** and **[DEGST90]**, and Professor Mladenov for useful conversations and for supplying us with the page proofs of **[M89]**. We would also like to thank Tad Wieczorek for correcting a number of errors in a preliminary version of this manuscript.

For background material in symplectic geometry we refer to our book **[GS84]** and in general relativity to the book by Kostant **[K88]**.

Apology. One of the key groups we will be using is the connected component of the identity of O(2, 4), a group which has four components. In contrast to standard usage, we will denote this group (and similar groups for other signatures) by SO(2, 4). So for us the symbol S will mean "connected component" rather than "determinant one." This is to avoid having to deal with unpleasant subscripts.

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Index

This book is based on the Colloquium Lectures presented by Shlomo Sternberg in 1990. The authors delve into the mysterious role that groups, especially Lie groups, play in revealing the laws of nature by focusing on the familiar example of Kepler motion: the motion of a planet under the attraction of the sun according to Kepler's laws. Newton realized that Kepler's second law—that equal areas are swept out in equal times—has to do with the fact that the force is directed radially to the sun. Kepler's second law is really the assertion of the conservation of angular momentum, reflecting the rotational symmetry of the system about the origin of the force. In today's language, we would say that the group $O(3)$ (the orthogonal group in three dimensions) is responsible for Kepler's second law. By the end of the nineteenth century, the inverse square law of attraction was seen to have $O(4)$ symmetry (where $O(4)$ acts on a portion of the six-dimensional phase space of the planet). Even larger groups have since been found to be involved in Kepler motion. In quantum mechanics, the example of Kepler motion manifests itself as the hydrogen atom. Exploring this circle of ideas, the first part of the book was written with the general mathematical reader in mind.

The remainder of the book is aimed at specialists. It begins with a demonstration that the Kepler problem and the hydrogen atom exhibit $O(4)$ symmetry and that the form of this symmetry determines the inverse square law in classical mechanics and the spectrum of the hydrogen atom in quantum mechanics. The space of regularized elliptical motions of the Kepler problem (also known as the Kepler manifold) plays a central role in this book. The last portion of the book studies the various cosmological models in this same conformal class (and having varying isometry groups) from the viewpoint of projective geometry. The computation of the hydrogen spectrum provides an illustration of the principle that enlarging the phase space can simplify the equations of motion in the classical setting and aid in the quantization problem in the quantum setting. The authors provide a short summary of the homological quantization of constraints and a list of recent applications to many interesting finite-dimensional settings. The book closes with an outline of Kostant's theory, in which a unitary representation is associated to the minimal nilpotent orbit of $SO(4,4)$ and in which electromagnetism and gravitation are unified in a Kaluza–Klein-type theory in six dimensions.

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