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Large Deviations

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Large Deviations

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Preface

These notes are based on a graduate course on large deviations given at the Courant Institute in 2012. While a version of these notes appeared on the web at that time, it took considerable time for me to prepare a revision. The lectures focused on three sets of examples as do these notes:

- diffusions with small noise and the exit problem,
- large time behavior of Markov processes and their connection to the Feynman-Kac formula and the related large-deviation behavior of the number of distinct sites visited by a random walk,
- interacting particle systems, their scaling limits, and large deviations from their expected limits.

We will look at simple exclusion processes in d dimensions. Some of the material is quite intricate and towards the end instead of providing complete proofs, we will give the ideas behind the proofs and provide references.

I want to thank the students who attended the course and motivated me to write these notes. It took much longer than I expected for me to finish the revision and I want to thank the AMS for waiting patiently. I want to thank Ina Mette, whose regular but gentle reminders prevented the delay from being even longer.

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Large Deviations

S. R. S. VARADHAN

The theory of large deviations deals with rates at which probabilities of certain events decay as a natural parameter in the problem varies. This book, which is based on a graduate course on large deviations at the Courant Institute, focuses on three concrete sets of examples: (i) diffusions with small noise and the exit problem, (ii) large time behavior of Markov processes and their connection to the Feynman-Kac formula and the related large deviation behavior of the number of distinct sites visited by a random walk, and (iii) interacting particle systems, their scaling limits, and large deviations from their expected limits. For the most part the examples are worked out in detail, and in the process the subject of large deviations is developed.

The book will give the reader a flavor of how large deviation theory can help in problems that are not posed directly in terms of large deviations. The reader is assumed to have some familiarity with probability, Markov processes, and interacting particle systems.

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