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JEROME K. PERCUS

LECTURE NOTES

Mathematical Methods in Immunology

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Preface

Back in the mid-1980s, I felt that immunology was an increasingly important field that might profit from appropriate mathematical analysis and made the obvious move: I developed and taught a course on the topic. It soon became clear that most of what I regarded as the valuable insights were due to one person: Alan Perelson, at Los Alamos, and so I contacted Alan, initiating a period of occasional collaboration that kept me informed of what had been discovered and what was being done about it. One result was that when I repeated this course every few years, it had the benefit of considerable new material. The last extensive rewriting was in 2002, and this, with embarrassingly few updates, represents the material presented in these lecture notes.

The topic coverage in a one-semester course on quantitative aspects of a large field must involve compromises and have a restricted scope. Here, this was accomplished by focusing in the main on the battle between the HIV virus and the adaptive immune system. This means that, on the biology side, the rapidly expanding study of the innate immune system and the role of inflammatory response are simply ignored, a dangerous but necessary tactic. The topics of allergy and autoimmunity—among others—are absent as well. And of course, the roles of organisms normally associated—largely internally—with that of the mammalian systems that are implicitly our focus, are not even mentioned. On the more mathematical side, much of the material presented is in the "classical" format of population dynamics of a well-mixed population—translating at once to chemical kinetics. However, the balance is partially redressed by a number of excursions into small and/or discrete populations, making initial contact with the ubiquity of fluctuations, driving much of current research.

It goes without saying that advantage was taken of the talents of a number of individuals—many of Alan Perelson's collaborators, as well as Ora Percus, who is responsible for a good deal of the content and for the elimination of numerous ambiguities and non-sequiturs. And Daisy Mojar-Calderon, who converted my illegible scrawlings into material of potential value.

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Mathematical Methods in Immunology

JEROME K. PERCUS

Any organism, to survive, must use a variety of defense mechanisms. A relatively recent evolutionary development is that of the adaptive immune system, carried to a quite sophisticated level by mammals. The complexity of this system calls for its encapsulation by mathematical models, and this book aims at the associated description and analysis. In the process, it introduces tools that should be in the armory of any current or aspiring applied mathematician, in the context of, arguably, the most effective system nature has devised to protect an organism from its manifold invisible enemies.

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