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# CBMS

Regional Conference Series in Mathematics

Number 98

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Alexander Varchenko



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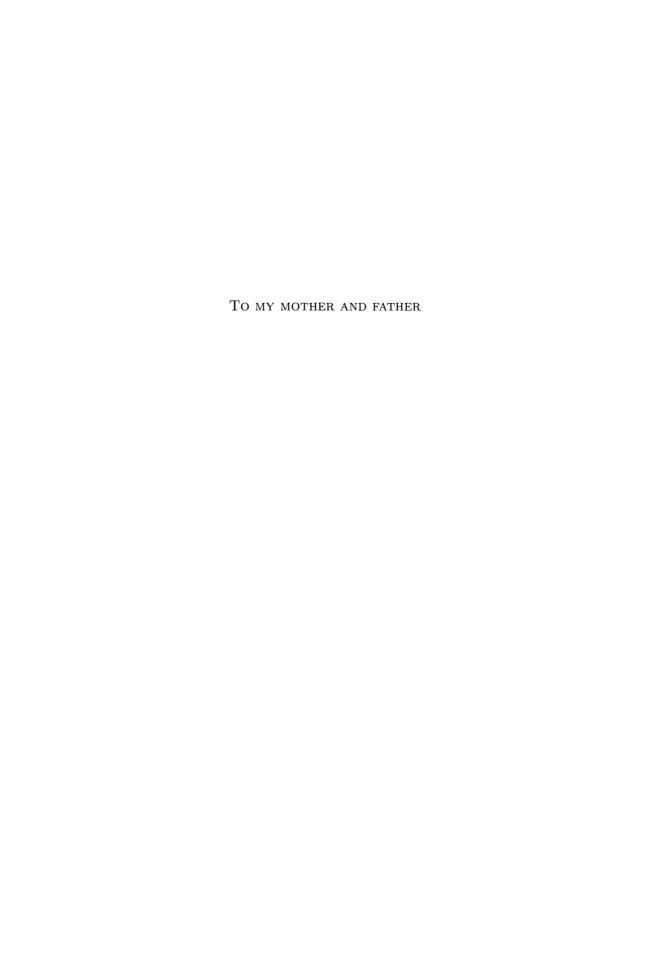
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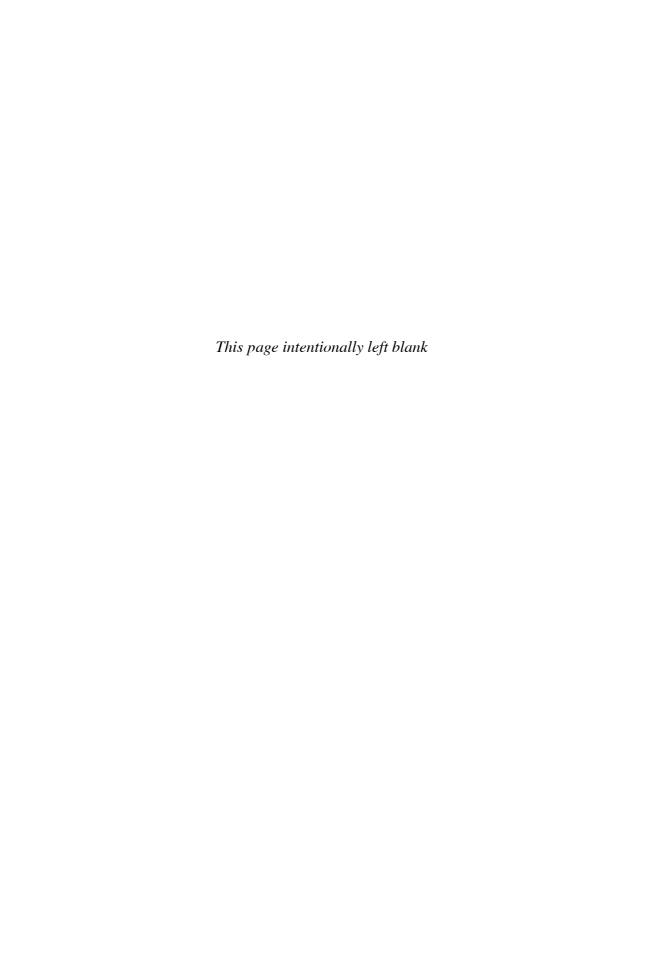
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## Contents

Preface		vii
Chapter	r 1. Hypergeometric Solutions of KZ Equations	1
1.1.	Examples of Hypergeometric Integrals	1
1.2.	Knizhnik-Zamolodchikov Equations	3
1.3.	Examples of Solutions	5
1.4.	Solutions of the KZ Equation with Values in Sing $M^{\otimes m}[ m -2k]$	6
1.5.	Solutions of the KZ Equation with Values in Sing $L^{\otimes m}[ m -2k]$	8
1.6.	The Classical Hypergeometric Series	8
1.7.	Identities for Differential Forms	10
1.8.	Hyperplane Arrangements	11
Chapter	r 2. Cycles of Integrals and the Monodromy of the KZ Equation	15
2.1.	Cycles for Solutions in Sing $M^{\otimes m} \left   m  - 2 \right $	15
2.2.	The Quantum Group $U_q(\mathfrak{sl}_2)$	18
2.3.	The Yang-Baxter Equation and Representations of Braid	
	Groups	20
2.4.	Quantum Singular Vectors	22
2.5.	The Monodromy of KZ Equations	23
2.6.	A Remark on Integration Cycles	29
Chapter	r 3. Selberg Integral, Determinant Formulas, and Dynamical Equations	31
3.1.	The Selberg Integral	31
3.2.	A Connection with Finite Reflection Groups	32
3.3.	An Example of a Determinant Formula	33
3.4.	The Determinant Formula for Solutions in a Weight Subspace	36
3.5.	General Determinant Formulas	38
3.6.	Resonances	40
3.7.	Dynamical Equations	43
Chapter	r 4. Critical Points of Master Functions and the Bethe Ansatz	49
4.1.	The Gaudin Model and the Bethe Ansatz	49
4.2.	Asymptotic Solutions and Eigenvectors	50
4.3.	Quasi-Classical Asymptotics of Solutions to the KZ Equation	53
4.4.	The Shapovalov Norm of Bethe Vectors	56

vi CONTENTS

4.5.	The Number of Critical Points of a Product of Powers of Linear Functions	58
4.6.	Critical Points of $\Phi_{k,n}(t,z,m)$ if $m_1,\ldots,m_n$ Are Natural	
4.5	Numbers	60
4.7.	Critical Points and Fuchsian Equations with Polynomial Solutions	62
4.8.	Resonant Local Systems	67
Chapter		69
5.1.	Knizhnik-Zamolodchikov-Bernard Equations	69
5.2.	The Case of $n = 1$ and $V = L_{2p}$	70
5.3.	Quasi-Classical Asymptotics of Solutions to the KZB Heat	71
F 1	Equation	71
5.4.	Elliptic Hypergeometric Functions Associated with One Marked	70
==	Point  Integral Depresentations for Elliptic Hymenson stric Eurotions	72
5.5. $5.6.$	Integral Representations for Elliptic Hypergeometric Functions Elliptic Selberg Integrals	$73 \\ 74$
5.0. 5.7.	Transformations Acting on the Space of Conformal Blocks	75
5.7. $5.8.$	Relations Between Theta Functions	76
5.9.	Basic Relations Between Elliptic Hypergeometric Functions	77
5.10.	Macdonald Polynomials and the Shift Operator	79
5.11.	Coefficients $f_{m,n}^{(k)}$ and Values of Macdonald Polynomials	80
5.11.	Modular Transformations of Elliptic Hypergeometric	80
0.12.	Functions	81
5.13.	Trace Functions for $U_a(sl_2)$	82
0.10.	Trace randoms for $\mathcal{C}_q(\mathcal{S}_{2})$	02
Chapter		85
6.1.	Quantum Knizhnik-Zamolodchikov Equations	85
6.2.	Quasi-Classical Asymptotics of Solutions and Eigenvectors	88
6.3.	An Example of Quantization of Hypergeometric Functions	89
6.4.	q-Hypergeometric Solutions, General Case	97
6.5.	The q-Hypergeometric Pairing	103
6.6.	Quantization of the Kohno-Drinfeld Theorem	105
Bibliography		109
Index		117

## **Preface**

The last twenty years have seen an active interaction between mathematics and physics. This book is devoted to one of the new areas which deals with mathematical structures related to conformal field theory and its q-deformations. We discuss the interplay between Knizhnik-Zamolodchikov type equations, the Bethe ansatz method, representation theory, and geometry of multi-dimensional hypergeometric integrals.

This book aims to provide an introduction to the area and expose different facets of the subject. The book contains constructions, discussions of notions, statements of theorems, and illustrating examples. The exposition is restricted to the simplest case of the theory associated with the Lie algebra  $\mathfrak{sl}_2$ .

The book consists of six chapters.

In Chapter 1 the Knizhnik-Zamolodchikov equation is realized as a Gauss-Manin connection, that is as the differential equation for multi-dimensional hypergeometric integrals.

In Chapter 2 the space of hypergeometric solutions of a Knizhnik-Zamolodchikov equation is identified with the multiplicity space of the tensor product of representations of the corresponding quantum group. This identification allows one to identify the monodromy group of the Knizhnik-Zamolodchikov equation with the R-matrix representation of the braid group.

In Chapter 3 determinant formulas for hypergeometric integrals are discussed. The determinant formulas allow one to conclude that the hypergeometric solutions give all solutions to the Knizhnik-Zamolodchikov equation. The second topic of Chapter 3 is the dynamical equations. The dynamical equations are equations complementary to the Knizhnik-Zamolodchikov equations. In the simplest example the equations respectively become the Gauss contiguous function relations and the Euler hypergeometric differential equation for the classical hypergeometric function.

In Chapter 4 the relations between the Bethe ansatz method and quasiclassical asymptotics of hypergeometric functions are discussed.

Chapter 5 is devoted to elliptic hypergeometric functions, their modular properties, and relations to Macdonald polynomials and traces of intertwining operators.

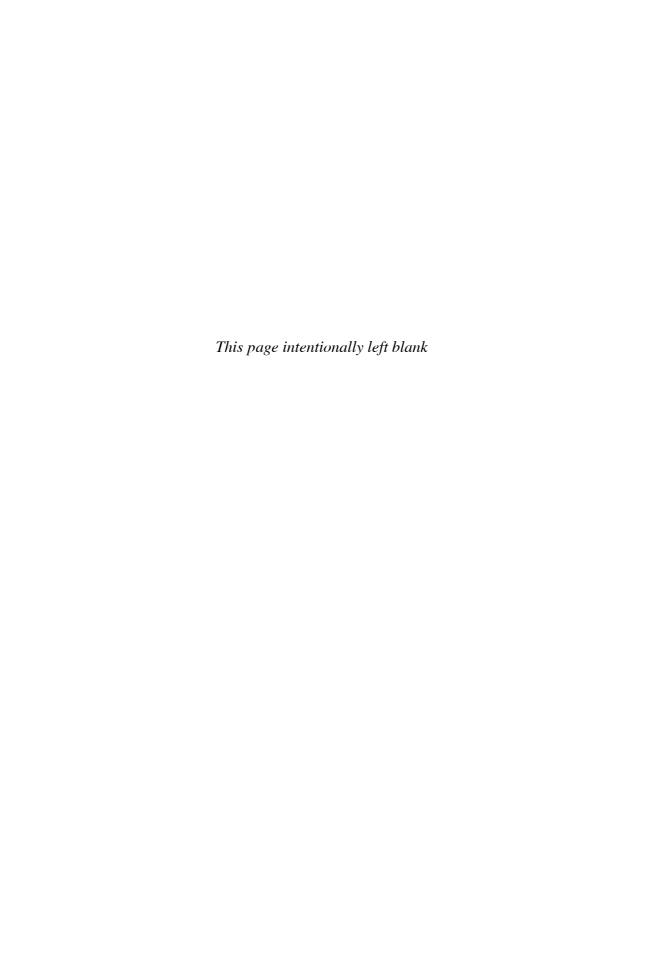
In Chapter 6 we explain how the geometric theory of the Knizhnik-Zamolodchikov differential equation has to be quantized to give solutions to the quantum Knizhnik-Zamolodchikov difference equation.

viii PREFACE

The book grew out of graduate courses at UNC in the fall of 2001, at MIT in the spring of 2002, and ten lectures given by the author at the NSF-CBMS Conference "Arrangements and Mathematics Physics", at Louisiana State University, Baton Rouge, during the week of January 11-15, 2002.

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Alexander Varchenko Chapel Hill, March of 2003



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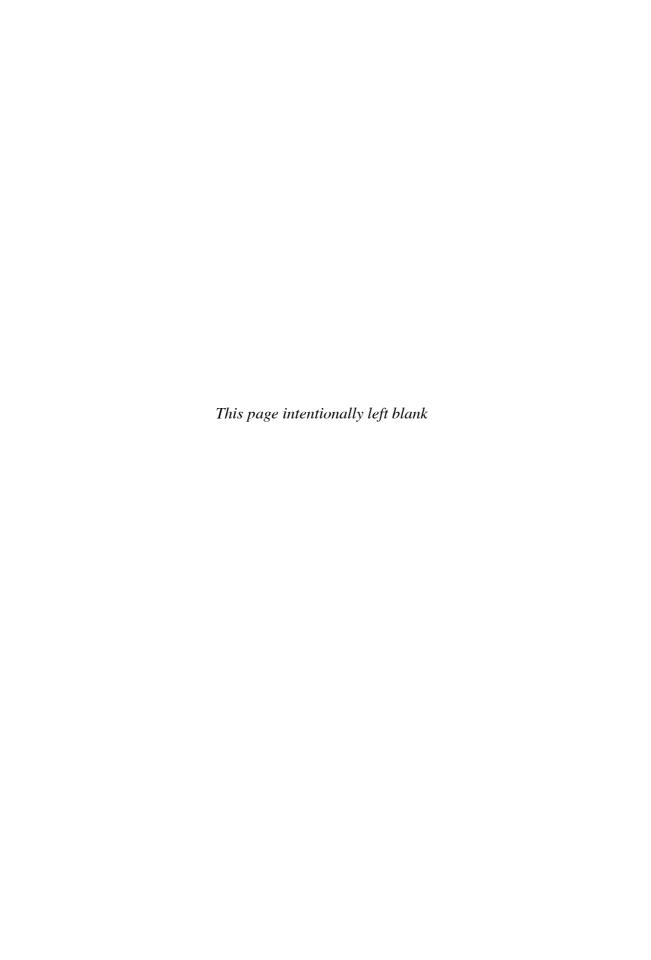
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#### Index

p-cycles, 91 hypergeometric functions associated with p-homology group, 91 a master function, 7 p-master function, 103 hypergeometric pairing, 104 hypergeometric solutions to KZ equation, amplitude, 53 asymptotic solution, 51, 88 hypergeometric solutions to trigonometasymptotic solution in an asymptotic zone, ric KZ equation, 45 indicial equation, 63 base points of a vector space of polynointegrals of the steepest descent method, mials, 66 53 Bethe ansatz equations, 50, 55 Jacobi polynomial, 64 Bethe ansatz method, 49, 50, 55, 72 Bethe ansatz vector, 50, 55 Knizhnik-Zamolodchikov equation, 3 braid group, 20 Lame equation, 72 conformal blocks, 41, 72 Lame function, 64 critical value of  $f^{\alpha}$  on a polygon, 37 Macdonald polynomials, 79 difference equations of the discrete Gaussmaster function, 6, 50 Manin connection, 93 modular group, 75 differential equation associated with a critical point, 64 non-degenerate space of polynomials, 66 discrete Gauss-Manin connection, 92 discriminantal arrangement, 26 phase, 53 dynamical equations, 43, 46 pure braid group, 20 elliptic Calogero-Moser system, 70 q-hypergeometric functions, 104 elliptic hypergeometric functions, 69, 72 qKZ equation, 85 elliptic master function, 70 qKZ operators, 87 elliptic Selberg integrals, 75 rational R-matrix, 86 exponents of a regular singular point, 63 real discriminantal arrangement, 29 flat sections of the discrete Gauss-Manin regular singular point, 62 connection, 93 scattering matrix, 106 Fuchsian differential equation, 63 Shapovalov form, 51 fundamental space of a critical point, 65 shift operator, 79 Gauss-Manin connection, 24 transition matrix, 106 Hamiltonians of the Gaudin model, 49 trigonometric classical r-matrix, 44 Hamiltonians of the XXX model, 88 trigonometric KZ equation, 44

118 INDEX

trigonometric KZ operators, 44 trigonometric R-matrix, 101

Weierstrass  $\wp\text{-function},\,70$ 

Yangian, 85

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