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Preface

These notes result from a meeting held in Santa Barbara in August 1983. The purpose of the meeting was to bring together geometric topologists and differential geometers to study in depth the recent work of Simon Donaldson. Of course, due to the beautiful and profound results of Mike Freedman, the subject of 4-manifolds had already become the focus of lively interest. Moreover, in light of Donaldson's result, the Freedman–Casson machinery was able to produce the startling fact that there exist exotic differentiable structures on $\mathbb{R}^4$. For these reasons topologists have wanted to understand in depth the arguments of Donaldson, which are based on the theory of Yang–Mills fields.

Consequently, the principal purpose of these lectures (and these notes) is to present these arguments together with all the background material required by someone who is not an expert in the field. The lectures are aimed, however, at mature mathematicians with some training in geometry and topology. The task set out here was already sufficiently exacting that no time was available for wide-ranging discussion or excursions into physics. On the other hand, this presentation attempts to be nearly complete.

It should be mentioned that a seminar on this subject was run last spring at M.S.R.I. by M. Freedman and K. Uhlenbeck. The notes of this seminar have been prepared for publication with the assistance of D. Freed. They also provide a detailed reference for this material.

The success of the conference was due to the enormous efforts of the organizing committee: Ken Millett, Doug Moore, and Marty Scharlemann, to whom all of us who participated have expressed our gratitude. I would also like to thank Susan Crofoot for her beautiful job of preparing the manuscript and the organizing committee for all the help they gave me with proofreading.

It is a pleasure to report that two of the conference participants, Ron Fintushel and Ron Stern, have subsequently succeeded in greatly generalizing the results of Donaldson while, at the same time, simplifying the arguments. They have also applied their methods to prove a spectacular result concerning homology cobordisms of homology 3-spheres. In particular, they show that the Poincaré 3-sphere has infinite order in this group. Interestingly, the key to their arguments is to consider gauge fields with SO_3 in place of SU_2. For low instanton numbers, the moduli space of self-dual connections in this case is actually *compact*. We shall say a bit more about this at the end of Chapter I.

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