Was Something Wrong with Beethoven's Metronome?

Sture Forsén, Harry B. Gray, L. K. Olof Lindgren, and Shirley B. Gray

Mathematicians and scientists seldom rest easy when the numbers are not right. The search for understanding is never far from mind. In many ways, this is why mathematics is so much fun. Similarly, conductors and musicians, who immediately recognize wrong notes, are perplexed by Beethoven's metronome markings. Some of his tempo markings, even on many of his most popular classics, have been considered so fast as to be impossible to play. What is the problem? Why?

The pianist and musicologist Peter Stadlen (1910–1996), who devoted many years to studies of Beethoven's markings, regarded sixty-six out of a total of 135 important markings as absurdly fast and thus possibly wrong [1], [2]. Indeed, many if not most of Beethoven's markings have been ignored by latter day conductors and recording artists. This situation is all the more puzzling since Beethoven himself was a strong proponent of the metronome and took the instrument to heart when it first came into his hands about 1816. He even expressed satisfaction that finally his intentions as to the tempi in his more important compositions would be made generally known.

The literature on the subject is enormous. However, this article will focus on the salient features of the early metronomes as seen through the eyes of a scientist, engineer, or mathematician. We investigate their early history, how they were constructed, and their mechanical properties. We investigate possible sources of error as to why Beethoven may not have been able to correctly note reliable and transferable time measures. We hope to demonstrate that there are possible mathematical explanations for the "curious" tempo markings—explanations that hitherto have not been considered except perhaps by Stadlen, who even went so far as to locate Beethoven's own metronome.

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Musical Notation and Tempo Indications in Beethoven's Lifetime

The musical notation used in European classical music—the five-line staff indicating pitch and notes of different appearance, whole notes, half notes, quarter notes, etc., to indicate the relative length of a tone within a time interval such as the "beat" or the "bar"—was essentially in widespread use by the end of the seventeenth century [3]. What was lacking in these early days was a universal way of indicating the tempo. *Relative* tempi could be used. There was the Italian system, still in use, with expressive words that run the gamut from fast to slow, proceeding from *prestissimo* all the way through *vivace*, *allegro*, to *adagio* and the sluggish *larghissimo*. There was, and is, also the German equivalent, again from fast to slow, going from *schnell*, perhaps preceded with an amplifying *sehr*, towards *sehr langsam*. Of course there is the French as well. The verbal way of marking tempi gives considerable freedom to the performing musician and thus a certain amount of artistic latitude to adapt to the current moment. Nevertheless, a number of composers, performing artists, and conductors were not satisfied with the lack of precision. In fact, already in the early years of the seventeenth century, the human pulse was used for timing. It is reported that the pulse was taken as eighty beats/minute—which seems somewhat high but may reflect the level of stress of performing artists. Also, the use of clocks was proposed by Henry Purcell in the late seventeenth century, and then there were devices based on the properties of simple pendulums.

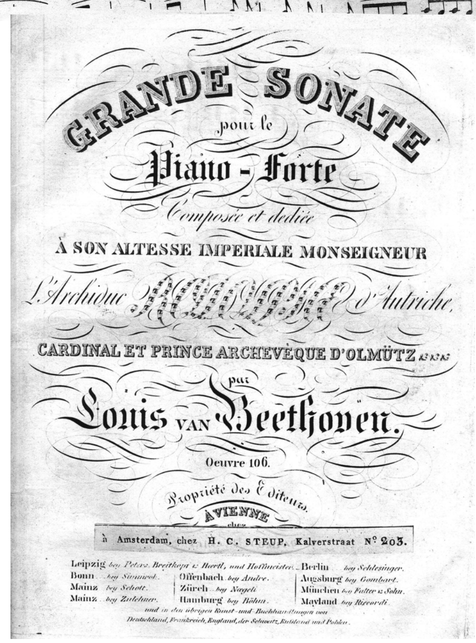
The study of the period of a simple pendulum is associated with the great Italian scientist Galileo. In Pisa he is thought to have experimented with a piece of string, fixed at one end and with a weight at the other. First, a pendulum is independent of the mass of the weight for a constant length of the string, and secondly, the square of the period of time varies directly with the length of the string. Galileo's results were eventually put into use for musical timekeeping. Perhaps the first was the French flutist and musicologist Etienne Louliè at the end of the seventeenth century. His device was rather unwieldy—it is reported to have been nearly two meters (approximately six feet) tall and obviously not very mobile. But his basic design was gradually refined by others who equipped a clockworks device to keep the pendulum swinging and also added a graduated scale by which the length of the string with the weight could be both altered and measured. A drawing of a pendulum instrument, i.e., the *metromètre*, designed ca. 1730 by Count D'Ons-en-Bray is reproduced in an article by Thomas Y. Levin [4]. Unwieldy as they must have been, the pendulum-based timekeeping instruments seem to have been in use well into the nineteenth century for want of a simpler alternative. Readers may want to compare the metromètre with Huygens's-pendulum clock (1673) [5].

But a "quantum jump" in the history of musical timekeeping was not far ahead. In Amsterdam in 1812 Dietrich Nikolaus Winkel (1780–1826) was experimenting with pendulums. He made the discovery that a special variant of a pendulum, in this case a thin wooden or metal beam some 15–20 cm in length and onto which weights could be attached, behaved in an interesting way. If the beam was able to swing freely around a pivot and two weights were attached to the beam, one on either side of the pivot, the pendulum would beat a steady time. Furthermore, with proper adjustment of the position and mass of the weights, it would beat either slow tempi, which previously had required very long single weight pendulums, or very fast tempi if the weights were moved toward the pivot. The principle of the metronome was born. Readers may picture the new invention as an augmented application of the Archimedean Law of the Lever.

Winkel must have realized the double pendulum principle could be applied to musical timekeeping. But there was obviously more to be done before the idea could be transformed into an easy-to-use instrument. Winkel spent a few years experimenting with his ideas. Finally, in August of 1815 he described his new invention in the *Reports of the Netherlands Academy of Sciences* where it also received extensive praise in a commentary. Though published, Winkel was soon to meet his nemesis. Winkel was no businessman for he had not patented his invention. He was soon approached by an enterprising man from Vienna who was something of a ruthless entrepreneur, a man named Mälzel.

The Multitalented Herr Mälzel

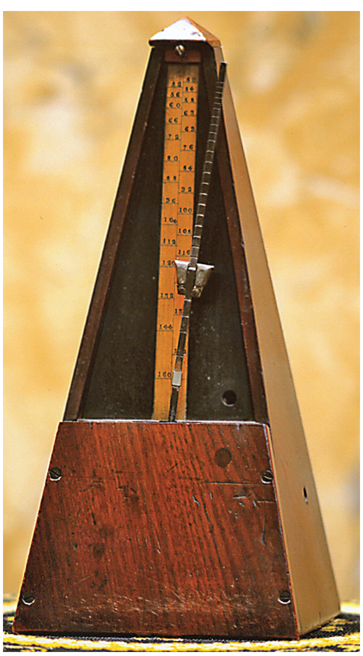
Courtesy of the Beethoven Center, California State University, San Jose.



Courtesy of the Beethoven Center, California State University, San Jose.

Johann Nepomuk Mälzel (1772–1838) was quite an interesting character. He was born in Regensburg where his father was a skillful organ builder. Young Johann turned out to have considerable musical talent. At the age of fourteen he was already regarded as one of the best piano players in the city. But he was also a most gifted mechanical engineer. At the age of twenty he established himself in Vienna and eventually gained access to a mechanical workshop in the famous Stein factory, producer of one of the pianoforte instruments that Beethoven favored. Mälzel was seemingly not without a well-developed taste for publicity. Moreover, his talent was in constructing musical automatons, devices that appealed greatly to the public at that time. A recent investigation reveals that he may in fact have copied inventions by others, changing them slightly and then claiming that they were his own. One of his most famous mechanical constructions was the remarkable *panharmonicon* that could imitate all instruments in a military band—even gun shots! It was actuated by a bellow. The notes played were determined by pins attached to a large rotating wooden cylinder, much like in old barrel-organs. The amazing “device” was demonstrated not only in Vienna but also in Paris where he sold his first instrument in 1807. It caught the eye of Luigi Cherubini who wrote a composition named *Echo* for the panharmonicon. It is indeed sad that the only copy of the panharmonicon, which for many years was kept at the *Landesgewerbemuseum* in Stuttgart, was destroyed in a bomb raid in 1942.

Mälzel’s reputation as somewhat of a mechanical wizard grew and in 1808 he received the title *Hofkammermaschinist*, a title that translates as court or royal machinist. He came in contact with Beethoven, who sought help for his increasing hearing loss. Mälzel constructed several ear trumpets for him, some still on display at the Beethoven Haus in Bonn. During Beethoven’s visit to Mälzel’s workshop, the problem of musical timekeeping was almost certainly discussed, and Mälzel apparently started working on the issue [6]. It appears that in 1813 he actually made some kind of timekeeping device, a “chronometer”, possibly based on an earlier design by G. E. Stöckler. In June of the same year Mälzel also suggested to Beethoven that he write a symphony to celebrate the Duke of Wellington’s recent victory over Napoleon’s troops at Vitoria in Spain, and he added that the composition should be arranged for the panharmonicon. Beethoven agreed and, as was his habit, outlined the composition. The collaboration ended in a bitter conflict between the two. Mälzel considered himself the rightful owner of the final work although the panharmonicon version was eventually abandoned and the composition rewritten by Beethoven for a symphony orchestra and first performed in this form in December of 1813. Beethoven dismissed Mälzel’s claims and instigated legal action against him.



Courtesy of the Beethoven Center, California State University, San Jose.

Chess players will be interested to know that in later life Mälzel purchased a remarkable chess playing automaton designed in 1770 by Baron Wolfgang von Kampelen. This automaton was generally known as “The Turk” after the dress of the almost life-sized doll that moved the chess pieces. Although seemingly run by an intricate mechanical assembly of cog-wheels and rods, it actually hid a human player in a way that escaped even the most inquisitive skeptics. Mälzel successfully toured with the Turk, who did beat most opponents, Napoleon among them, all over Europe. Later he toured the United States where, however, the human involvement inside the Turk was accidentally exposed. Mälzel, who became a wealthy man, died in 1838 of an overdose of alcohol on a ship in the harbor of LaGuaira, Venezuela. The Turk eventually ended its days in 1854 when it was destroyed by fire at the National Theater in Philadelphia. Both Wikipedia and the magician James Randi have described how the public was fooled into accepting the idea that no human could possibly be hidden inside [7].

But now back to the invention of the double pendulum by Dietrich Winkel in Amsterdam. News traveled fast even in the early part of the nineteenth century. Mälzel had obtained some information on Winkel’s invention as early as 1812. It is even possible that the two had briefly met that year. But after the publication of Winkel’s invention in August of 1815, Mälzel hurried to Amsterdam, met with Winkel, inspected the new “metronome” and realized its superiority over his own timekeeping devices. He offered Winkel money to buy the right to sell the device under his own name. Not surprisingly, Winkel refused.

Intellectual property rights were rarely enforced in those days, so Mälzel went back to Vienna, made a copy of Winkel’s instrument, added a graded scale to the oscillating beam on the side of the movable weight, took the copy to Paris, and saw to it that “his”—Mälzel’s—invention was patented there and later also in London and Vienna. He even set up a small factory, Mälzel & Cie, in Paris for the production of “his” metronomes.

In 1817 Winkel became fully aware of Mälzel’s activities and he was understandably upset. He instigated proceedings against Mälzel for the obvious theft. The Netherlands Academy of Sciences was involved as arbitrator and ruled in Winkel’s favor. To no avail, Winkel had been scooped. The metronome factory in Paris was up and running. Metronomes soon became very popular. Mälzel was regarded as the true inventor and the abbreviation “MM” for “Mälzel’s Metronome” was commonly placed before metronome measures in printed sheets of music.

Beethoven and Mälzel's Metronome

The first metronomes based on Winkel's double pendulum principles from Mälzel's Paris factory should have become available in early 1816. It is thus possible that Beethoven was presented with a copy around this time, perhaps as a gesture of peace from Mälzel, who must have realized that Beethoven's approval of the instrument would be good for business. Not much is mentioned about the metronome in Beethoven's letters in 1816, perhaps due to the master's preoccupation with pressing family matters. His brother Caspar Carl died of tuberculosis in November 1815, leaving behind a nine-year-old son, Karl. Beethoven made every effort to become the sole guardian of his young nephew, to the point of taking the case to court in opposition to Karl's mother, Johanna. The legal battle went on for more than four years, causing considerable emotional strain on young Karl [8].

It is reported that in 1816 Beethoven was so preoccupied by these legal dealings that he stopped composing. But from 1817 onward the metronome was indeed on his mind. At that time he wrote Hofrath von Mosel, "So far as I am myself concerned, I have long purposed giving up those inconsistent terms 'allegro', 'andante', 'adagio', and 'presto'; and Mälzel's metronome furnishes us with the best opportunity of doing so" [9]. Later, his ninth symphony, in addition to his earlier symphonies, was marked with metronome measures. After a report from a performance of its popularity in Berlin, he wrote to his publisher Schott, "I have received letters from Berlin informing me that the first performance of the ninth symphony was received with enthusiastic applause, which I attribute largely to the metronome markings" [10]. Surprisingly, Beethoven only gave metronome markings to one of his piano sonatas, Op. 106, also called the *Hammerklavier sonata*. Here the first movement, an allegro, is marked one hundred thirty-eight beats per minute for the half note, which is extraordinarily fast. The great pianist Wilhelm Kempff made some rather harsh comments on this marking in his set of recordings of the later piano sonatas, "The erroneous (sic!) metronome markings can easily lead to this regal movement being robbed of its radiant majesty."

Assuming Beethoven had one of Mälzel's metronomes in his possession by 1817, he probably could not hear the "clicks" of the repetitive beat as he was suffering from approaching deafness at that time. But, of course, he could *see* the oscillating beam. So what could possibly have caused him to indicate the fast tempi throughout that so puzzled subsequent generations of musicians? Have present-day musicians tried to make his compositions more "romantic" than the master intended or else given up as being technically too difficult?

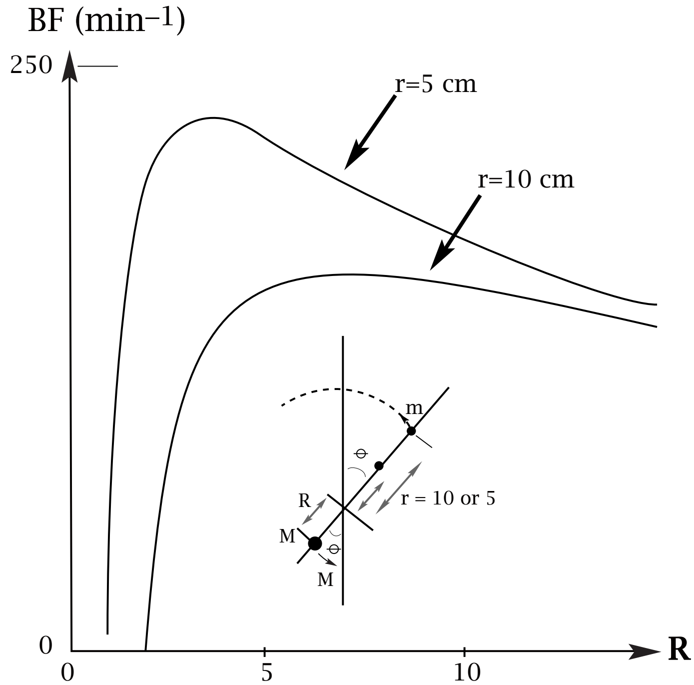


Figure 1. The beat frequency, BF, of a metronome as a function of the distance R of the heavy weight (M=40 g) from the pivot axis at two fixed positions of the movable weight: r= 5 cm and r= 10 cm. As is evident, the beat frequency will be extremely sensitive to a small change in the position of the heavy weight—in particular for small values of r.

The beat rate was indicated by gradations on the oscillating beam. Beethoven's eyesight was not the best. He could have read the wrong numbers. On the other hand, he often had help from his nephew Karl, who may not have had vision problems. Then there is the connection between the beat rate and the note length: a beat rate of "60" could refer to the length of a whole note, a half note, or a quarter note, and the tempo of the music changes accordingly. A beat rate of 60 for a whole note becomes 120 for a half note. Accidentally removing the stem from the sign of the half note would seemingly double the tempo. Could copy errors have been the culprit?

We should also consider psychological factors. Different humans may have slightly different "internal clocks". A tempo regarded as fast for one person may be less so for another. We suppose that internal clocks have a tendency to run slower with age in most humans, one notable exception being Toscanini's, whose clock seemed famously to never lose time.

But one must also consider reports that Beethoven's metronome on occasion was out of

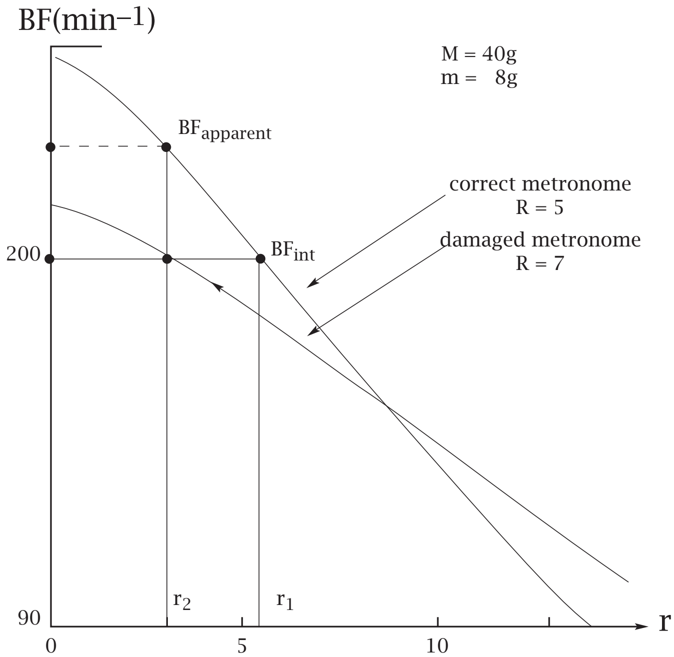


Figure 2. The beat frequency, BF, of a metronome as a function of the distance r from the pivot axis of the movable weight (m = 8 g). The heavy weight (M= 40 g) is assumed to be at the distance R= 5 cm from the pivot axis for the correct, factory-calibrated metronome and at the distance R= 7 cm for the damaged metronome. The damaged metronome will beat slower than indicated by the factory-engraved markings for a calibrated and correct metronome when r is less than 8 cm, while it will go faster than the engraved markings at r larger than 8 cm. Thus when r < 8 cm, $BF_{damaged} < BF_{correct}$; when r = 8 cm, $BF_{damaged} = BF_{correct}$; when r > 8 cm, $BF_{damaged} > BF_{correct}$.

order. In an April 1819 letter to his friend and copyist Ferdinand Ries at the Fitzwilliam Museum in Cambridge, Beethoven states that he cannot yet send Ries the tempi for his sonata Op. 106 *because his metronome is broken* [11]. Beethoven was very irritated and upset, perhaps from symptoms of lead poisoning, a condition he may have had according to studies at the Argonne National Laboratory [12]. Could Beethoven on occasion have dropped the metronome on the floor or could he have used more than one metronome in his last ten years?

Peter Stadlen is reported to have found Beethoven's metronome after a long search. This particular specimen was among the property auctioned off by his surviving nephew Karl after his uncle's death. To Stadlen's disappointment the heavy weight was gone, perhaps an indication that the early metronomes from Mälzel's factory did not have rugged, shock-proof construction.

Dynamical Properties of the Metronome

Finally, we take a closer look at the dynamical properties of the mechanical double pendulum metronome and, in particular, determine if its performance is sensitive to the position of the supposedly fixed heavy weight with respect to the pivot. (See the text below on the derivation of the equations of motion.)

The final result for the oscillation frequency Ω of a Winkel-type double pendulum is as follows:

$$\Omega = \left[\frac{g(M\mathrm{R} - m\mathrm{r})}{(M\mathrm{R}^2 + m\mathrm{r}^2)} \right]^{1/2}. \quad (1)$$

Here M is the mass of the heavier fixed weight beneath the pivot along with its distance R to the pivot axis. Correspondingly, m is the mass of the movable weight at distance r above the pivot point. The lighter weight and its distance are what musicians can see, while the heavier weight is usually hidden in the base of the cabinet. Also, g is the acceleration due to gravity (9.81 m/sec^2). But Ω is not the same as the beat frequency in "clicks per minute" of the metronome (**BF**). As can be seen in the derivation, the beat frequency **BF** is related to Ω by the ratio

$$\mathbf{BF} = 60\Omega/\pi. \quad (2)$$

The equations are valid under the assumption that there is no friction or drag, the masses are point masses, and the amplitude of the oscillations around the vertical position is small. The last assumption is not severe as the beat frequency would not deviate much from the above expression for relatively large amplitudes; however, the correct equations would be far more complex and nonlinear. Please note that **BF** is the limit when r goes to zero, and it is independent of the mass of the two weights as determined by R, or rather $1/\sqrt{\mathrm{R}}$, which is a consequence of m being located on the pivot axis with the double pendulum beating like a simple pendulum.

Equation (1) gives us an opportunity to investigate the sensitivity of the metronome frequency to a change in parameters. For any particular metronome we can assume that both M and m are fixed at the factory by the manufacturer. The value of R for the heavy weight is also likely to be set at the factory in order to make the metronome beat at the correct frequency when the moving weight is put at the corresponding grading on the oscillating beam. But what if an error had been made? Perhaps the heavy weight was attached to the beam in such a way that it actually could slide—move—from its original position. Suppose the metronome accidentally fell to the floor or was otherwise damaged. We could investigate what the consequences would be. But first, just for curiosity's sake, let us look at how the beat frequency of a metronome in clicks per minute, **BF**, will depend on the position of the **heavy weight** in relationship to the lighter weight at a fixed distance. In other words, suppose only the location of the heavy weight *M* is changed.

We now turn to numerical results. Although we have searched extensively, we have not found the actual values of the parameters *M*, *m*, and R that Beethoven used to obtain his tempo markings. Over a limited parametric space, however, the results are not sensitive to the values picked. For the sake of argument, let's assume $M = 40$ grams at distance R = 5 cm for the heavier weight and $m = 8$ at distance r = 10 for the upper weight. (See Figure 1.) We note that the beat frequency **BF** of a metronome is extremely sensitive to the position of the heavy weight if the distance R is less than about 3–4 cm and **BF** goes to zero at R = 1 cm and r = 5 cm. At the latter distances the two weights exactly balance each other with the cross product being (40)(1) = (8)(5). Under these conditions the metronome beam would rotate freely like a balanced propeller if there were no physical constraints. For r =10 this condition is met when R = 2; i.e., (40)(2) = (8)(10). As illustrated in Figure 1, the heavy weight should not be placed too close to the pivot as it is important to minimize the effects of small changes in its position. But it is also evident that, for R values larger than those at the maxima in the arcs of the curves, shifts in the position of the heavy weight toward larger distances from the pivot will result in **BF** shifts to smaller values. This effect is particularly striking when the movable weight is close to the pivot, i.e., in the region of fast tempi of the metronome!

Finally, let us look at what would happen if the metronome were somehow damaged by a shift in the heavy weight from its calibrated factory position. We consider two cases. First we will assume that the heavy weight is moved such that it is a longer distance ($R_{Beethoven}$) from the pivot, a situation that could occur if the metronome fell to the floor in an upright position. (We hope that Herr Ludwig, as he rests in heaven, will forgive us for the notation ($R_{Beethoven}$).) If we take ($R_{Beethoven}$) to be 7 cm and assume that the original factory position was 5 cm, the result is shown in Figure 2. For fast tempi (**BF** axis), the damaged metronome will go **slower** than indicated by the factory calibration. At a medium value of r we reach a point where the factory gradations happen to be correct, but for slow tempi (see R) the damaged metronome always will go **faster** than predicted by calibrations.

In the second case let us assume the heavy weight is moved to a position closer to the pivot than its original factory location, perhaps because the metronome had accidentally fallen to the floor with its top down. In Figure 3 the results are displayed for a case in which the factory position

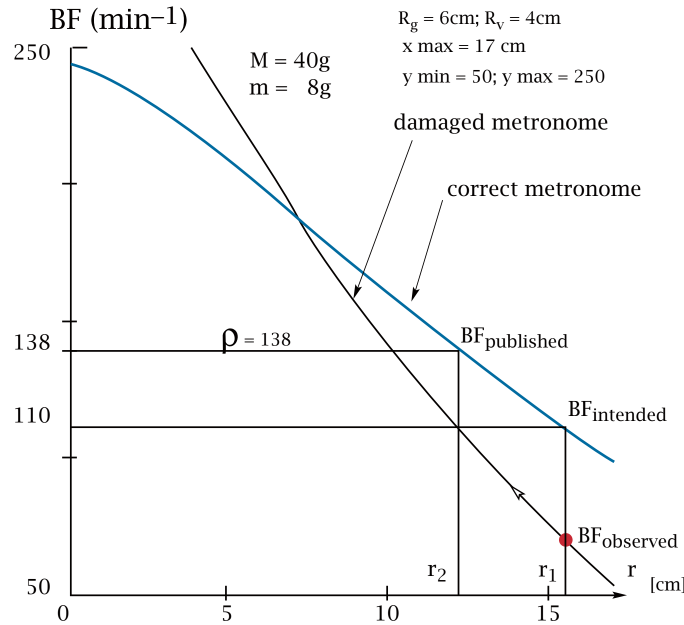


Figure 3. The beat frequency, BF, of a metronome as a function of the distance r from the pivot axis of the movable weight (m = 8 g). The heavy weight (M = 40 g) is assumed to be at the distance, R= 6 cm from the pivot axis for the correct, factory-calibrated metronome and at the distance R= 4 cm for the damaged metronome. Beethoven puts the movable weight on his metronome to correspond to the marking BF =110 (i.e., BF_{int} in Figure 3) but, somewhat puzzled perhaps, finds that the visibly observed BF seems far too slow, around seventy to eighty or so, i.e., the BF of the damaged metronome curve at the setting $\mathbf{r_1}$ (the point indicated by BF_{obs}.]

of the heavy weight is 6 cm from the pivot (R = 6 cm) and the distance after an accident is R = 4 cm. We note by looking at Figure 1 that, in the region of small values of R, even a small change in position will have a major effect on the metronome's beat frequency. Other sets of parameters produce even greater changes.

Could these results explain Beethoven's "absurdly fast" metronome markings? We should note that, if the same model is applied in the region of slow tempi (see Figure 2), the printed markings would result in somewhat **slower** beat frequencies on the damaged metronome than intended by the master.

Let's envision the following *hypothetical scenario* (cf. Figures 1–3). Unknown to him, the metronome Beethoven is working with is damaged in the sense that the heavy weight hidden by the wooden case has been displaced. Assume Beethoven puts the movable weight on his metronome to correspond to the marking of approximately **BF** =110 (i.e.,

BF$_{\text{intended}}$ in Figure 3). Somewhat puzzled perhaps, he finds the visibly observed **BF** seems far too slow, around 70 to 80, i.e., the **BF** of the damaged metronome curve at the setting r_1 (the point indicated as **BF**$_{\text{observed}}$). The markings on the metronome beam with the light movable weight that he can clearly see **do not** correspond to his desired **BF**. Beethoven, dissatisfied with the slow movement of the visible metronome beam, then moves the weight until he is satisfied with the much higher **BF**.

Today we know very little about who actually helped the master with the practical details of proceeding from his raw, hardly legible, handwritten sheets of music, but we are very sure that a **BF** = 138 was published (*p*=138). (See Figure 3.) A latter day pianist intending to perform the *Hammerklavier* sonata looks at the printed sheet of music marked a "half-note = 138," sets a correctly calibrated metronome, and mutters, "incredibly fast"—very fast indeed! Our *scenario* provides an explanation for Beethoven's "fast tempi problem", if indeed there is one.

How could Beethoven not note the occasional odd behavior of his metronome? A thorough account by Peter Stadlen gives the impression that the master was not entirely comfortable with the new device, most especially in the process of converting from beat frequencies to actual tempi markings for half-notes, quarter-notes, etc. [13]. Obviously, it would be very helpful if we knew more about the actual design of his metronome(s). We suggest that one or more of the devices could have been damaged, perhaps accidentally during one of his well-known violent temper tantrums. Whatever the case, our mathematical analysis shows that a damaged double pendulum metronome could indeed yield tempi consistent with Beethoven's markings.

Arguably this is a bold hypothesis. Perhaps someone else was involved in the procedure—Beethoven's eyesight was not always the best.

Derivation: Equations of Motion for a Double Pendulum for the Type Used in Mechanical Metronomes.

Consider the following model of the double pendulum:

We will assume that the beam oscillates around the pivot without any friction and also that the mass of the beam can be ignored in comparison to that of the two weights. The total energy E of the oscillating pendulum is a sum of its potential energy V and its kinetic energy T.

The kinetic energy may be written

$$T = \frac{1}{2}m(v_m)^2 + \frac{1}{2}M(v_M)^2 \tag{3}$$

where v_m and v_M are the velocity of the light and heavy masses, respectively, in the direction of their motions. Now

$$v_m = \frac{d(S_m)}{dt} \quad \text{where } (s_m) = \mathrm{r}\sin\Theta,$$
$$v_M = \frac{d(S_M)}{dt} \quad \text{where } (s_M) = \mathrm{R}\sin\Theta.$$

For small values of Θ we have $\sin\Theta = \Theta$ (in radians). Thus

$$s_m = \mathrm{r}\Theta \quad \text{and} \quad s_M = \mathrm{R}\Theta.$$

The velocities v are then

$$v_m = \mathrm{r}\frac{d\Theta}{dt} = \mathrm{r}\dot{\Theta} \quad \text{and} \quad (v_m)^2 = \mathrm{r}^2\dot{\Theta}^2,$$
$$v_M = R\frac{d\Theta}{dt} = \mathrm{R}\dot{\Theta} \quad \text{and} \quad (v_M)^2 = \mathrm{R}^2\dot{\Theta}^2.$$

The kinetic energy of the double pendulum is thus

$$T = \frac{1}{2}\left[m\mathrm{r}^2\dot{\Theta}^2 + M\mathrm{R}^2\dot{\Theta}^2\right]. \tag{4}$$

Now let us consider the potential energy V with respect to the axis of the pivot:

$$V = gm\mathrm{r}\cos\Theta - gM\mathrm{R}\cos\Theta, \tag{5}$$

where g is the acceleration due to gravity. Again assuming small angles of oscillation when $\sin\Theta = \Theta$ and since $\sin^2\Theta + \cos^2\Theta = 1$, we can replace $\cos\Theta$ with $(1-\Theta^2)^{\frac{1}{2}}$. The potential energy V then becomes

$$V = g[m\mathrm{r} - M\mathrm{R}](1-\Theta^2)^{\frac{1}{2}}. \tag{6}$$

But, if the double pendulum moves without friction, then the total energy E is constant and the time derivative will be zero:

$$\frac{dE}{dt} = 0.$$

Let us for the sake of clarity take the time derivatives of T and V separately:

$$\begin{aligned}\frac{dT}{dt} &= \frac{1}{2}\left[m\mathrm{r}^2 2]\dot{\Theta}\ddot{\Theta} + M\mathrm{R}^2 2\dot{\Theta}\ddot{\Theta}\right] \\ &= \left[m\mathrm{r}^2 + M\mathrm{R}^2\right]\dot{\Theta}\ddot{\Theta}\end{aligned} \tag{7}$$

where $\ddot{\Theta} = \frac{d^2\Theta}{dt^2}$.

Now consider the time derivative of the potential energy for small oscillations:

$$\frac{dV}{dt} = \frac{d}{dt}[g[m\mathrm{r} - M\mathrm{R}](1-\Theta^2)^{\frac{1}{2}}].$$

Note that

$$\frac{d\sqrt{(1-\Theta^2}}{dt} = \frac{1}{2}\left(\frac{-2\Theta\dot{\Theta}}{\sqrt{(1-\Theta^2)}}\right) \approx -\Theta\dot{\Theta} \text{ for small } \Theta.$$

Continuing,

$$\frac{dV}{dt} = g(M\mathrm{R} - m\mathrm{r})\Theta\dot{\Theta},$$

and since $\frac{dE}{dt} = \frac{d}{dt}[T+V] = 0$,

$$(mr^2 + MR^2)\dot{\Theta}\ddot{\Theta} + g(M\mathrm{R} - m\mathrm{r})\Theta\dot{\Theta} = 0. \tag{8}$$

We can eliminate $\dot{\Theta}$ since $\dot{\Theta} = 0$ does not give anything, and we are left with

$$\ddot{\Theta} + g\left[\frac{(M\mathrm{R} - m\mathrm{r})}{(m\mathrm{r}^2 + M\mathrm{R}^2)}\right]\Theta = 0. \tag{9}$$

This is a differential equation of the type

$$\frac{d^2x}{dt^2} + a^2x = 0$$

with the solution

$$x(t) = (x_0)\cos(ax + a),$$

and if we introduce

$$\Omega = \left[\frac{g(M\mathrm{R} - m\mathrm{r})}{M\mathrm{R}^2 + m\mathrm{r}^2)}\right]^{\frac{1}{2}},$$

the solution to equation (9) is

$$(\Theta(t) = (\Theta_0)\cos(\Omega t + \Omega).$$

Let us check to see if this expression yields the same for a simple pendulum if we put $m = 0$. We have

$$\Omega = \left[g\left(\frac{M\mathrm{R}}{M\mathrm{R}^2}\right)\right]^{\frac{1}{2}} = \left[\frac{g}{\mathrm{R}}\right]^{\frac{1}{2}},$$

$\Theta(t) = \cos[(g/\mathrm{R})t]$, and thus the oscillation frequency is $\Omega = (g/\mathrm{R})$, or the correct equation for small amplitude oscillations. We note that Ω is in fact the oscillation also for our double pendulum!

How do we relate Ω to the beat frequency of the metronome? The period of any type of pendulum is the time it takes to swing from one starting position and back to that position again. But this corresponds to **two "clicks"** of the metronome—it clicks at every turning point of the oscillating beam! Therefore, to go from pendulum period ($P = 2\pi/\Omega$) to beat frequency "**BF**" in "clicks per minute" we have

$$\mathbf{BF} = \mathbf{60}\Omega/\pi.$$

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About the Cover

Baltimore, MD, is the location of the 2014 Joint Mathematics Meetings. Pictured on the cover is a nighttime view of the Baltimore Inner Harbor.

One item of interest in Baltimore for mathematicians is that the Walters Museum was the site of work on the Archimedes Codex. In response to a query of ours, however, Nancy Zinn, deputy director at the Museum, reports that the codex is no longer there. She says further:

"Since the completion of our analysis and conservation of the manuscript, the codex has been returned to its owner, and is no longer on deposit at the museum. However … the owner has agreed to our request for the palimpsest to be on view at the Huntington Library in San Marino, CA, from March 15 to June 8, 2014. This will be the third, and final, venue for the exhibition 'Lost and Found: The Secrets of Archimedes'. It is unlikely that the codex will ever be displayed publicly again."

A complete report about the palimpsest can be found at

http://archimedespalimpsest.org/

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