Can One Hear the Sound of a Theorem?

Rob Schneiderman

Mathematics and music have been intertwined in a long-running drama that stretches back to ancient times and has featured contributions from many great minds, including Pythagoras, Euclid, Mersenne, Descartes, Galileo, Euler, Helmholtz, and many others (see, e.g., [1]). Applications of mathematics to music continue to develop in today's digital world, which also supports active communities of musicologists and experimental composers who examine music methodically, often using mathematical elements. In light of the recent wave of musico-mathematical books, blogs, journals, and even articles in the *Notices*, this multifaceted side of the mathematical world deserves reexamination. Although the scrutiny given here will reveal many problems posing as solutions, some promising prospects will also emerge, and positive turns in the plot may yet unfold, especially when viewed from a novel educational angle described below.

From the mathematician's perspective, besides providing a bounty of physical applications, the search for relationships between music and mathematics should serve both as a philosophical reflection pool and as a portal to an engagement of the general public with mathematics. But the view is often obstructed by the unwitting entanglement of several distinct lines of thought. It is not uncommon for commentary on music and mathematics to bounce between the physics of sound, theoretical analysis of music, and metaphorical prose. While each approach has its strengths and weaknesses, unjustified juxtapositions can serve to cloud the big picture by masquerading as implicit unifications of unresolved key issues or by appearing to support pseudoscientific arguments. For example, [4] and [5] exposit useful mathematical techniques in the setting of digital audio processing, which are then associated with flawed musical analysis and exaggerated conclusions. A historical article on the mathematics of fretting a guitar in [6] is presented side-by-side with musical numerology in a collection whose introduction enthusiastically includes as evidence of connections between mathematics and music the "ordering by number" of Bach's Goldberg variations!

As one who came to mathematics after a career as a professional musician, I offer here a personal viewpoint in hopes that it will provide a helpful framework for unwinding the current strands of a fascinatingly elusive subject. This essay will argue that while mathematics provides satisfying analyses of sound and useful parameterizations of musical choices, deeper scientific relationships between mathematics and music remain largely beyond reach. But the adoption of a more metaphorical point of view will uncover support for a return of music and mathematics to a quadrivium-like partnership in education that is based on a common strength of intrinsic structure.

The goal here is not to give a survey of the present state of musico-mathematical affairs but rather to highlight a representative sample of points that seem to be overlooked or underappreciated in the current general discourse. Of course personal taste enters into any discussion of music, and many issues raised below are subject to differing interpretations. The arguments are mostly critical because such objections seem to have had trouble finding their way into print, but I support many aspects of even the approaches criticized here and hope to clarify and stimulate the ongoing dialogue. It is in the interest of the mathematics community to engage in and be aware

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of the development of interdisciplinary work in all directions.

The body of this article is roughly divided into the subtopics of *science of sound*, *analysis of music*, and *metaphorical comparisons.*

The Science of Sound

There is much solid and fascinating mathematical work, classical and ongoing, which is related to musical sound, including instrument design, acoustics, and audio processing, among many interrelated topics. Applications of mathematics are readily apparent in the modern recording studio, where the signal of digitally recorded instruments (both electric and acoustic) is routinely manipulated in a wide variety of ways, including the independent adjustment of tempo and pitch of individual voices, as well as the elimination of ambient noise and creation of audio effects. Fourier theory plays a central role throughout these settings, essentially due to the periodic nature of musical sound waves and the graded elasticity of our ears' basilar membranes, which act as harmonic analyzers. (A broad introduction to the mathematics of musical sound can be found in the first eight chapters of David Benson's book [2].)

While elements such as rhythm, melody, and harmony are frequently described as fundamental "dimensions" of music, the case can be made that in fact *timbre* (or *tone color*) is the most important universal musical quality: The strike of Pablo Casals' bow to a cello string can send chills up the spine, and Nat Cole's voice can convert a single syllable into the sublime. In this case Fourier theory provides a very strong mathematical explanation for this musical phenomenon: namely, that the timbre of a sound—which closely corresponds to the frequency spectrum of its wave shape—lives in an infinite-dimensional space! Well, infinite dimensional in principle, but even taking into account the limited frequency range of our conscious hearing (20Hz–20,000Hz), just a single second of reasonably digitized musical sound will require tens of thousands of coordinates, as even the very short-term time evolution of wave shape is critical to the perception of tone quality. The depth and complexity of timbre is further illustrated by the extreme difficulty of synthesizing musically interesting sounds by directly prescribing wave spectra and by the fact that a pure sine wave corresponds to a completely boring musical sound.

Of course almost all musicians remain blissfully unaware of the elegance of Fourier theory as they coax out expressively complex sounds from traditional instruments, guided only by the analysis provided by their own ears. While it is true that, by electronically synthesizing "unnatural" spectra, it is possible to generate sounds that cannot be made by traditional instruments—perhaps following a musical analogue of studying nonstandard axiomatic systems in mathematics—such variations are not ends in themselves and have value only if they lead to "interesting" results.

Although timbre is fundamental to music, extending musico-mathematical relationships becomes problematic as sequences of sounds are extended in time and begin to acquire musical meaning. For instance, the well-studied relationships between whole number ratios and consonant pitch intervals, while interesting from a physical point of view and historically important, ultimately do not correspond to any cohesive *mathematical* notion, as musical esthetics rightfully leads to compromises and approximations in choices of scales and tunings, with the resulting widely accepted equal-tempered chromatic scale having frequency ratios of $\sqrt[12]{2}$ for all pairs of adjacent notes. Explanations of this are readily available, for instance in Chapter 5 of [2], as well as in Ian Stewart's delightful expository piece in Chapter 4 of [6], describing how a classical construction for placing the frets on a guitar ties together discussions of Pythagorean and equal-tempered scales, ruler and compass constructions, continued fractions, and fractional linear approximations of exponential functions. Although there does exist a small minority of musicians who are obsessed with subtleties of tuning choices and justifications of scale constructions, the vast majority of musicians have no trouble making beautiful music with the equal-tempered pitch system, easily incorporating together instruments having fixed tunings with those that are more flexible and happily exploiting the freedom to modulate between unrelated keys that is afforded by "theoretically compromised" scales. In any event, many instruments are tuned by hand, and notes are bent by ear, so it is not surprising that once musical flow commences, mathematical imperfections in pitch fade into the background.

Perhaps the irony that ancient hopes for combining rational numbers and music into a cohesive world view have been dashed by the general acceptance of a musical system based on $\sqrt[12]{2}$ is an omen representative of problems that will haunt future attempts to build bridges between mathematics and music.

Analysis of Music

Three overlapping goals of music theory are to *explain why* music sounds the way it does, find good *ways to listen* to music, and describe *how to create* music. What might mathematics have to do with these goals? It certainly is natural to use permutations and transformations in describing *available* musical choices and relations between them (for instance, by representing pitch/rhythm in frequency/time coordinates or numbering scale tones relative to a root). But attempts at exhibiting substantial connections between *meaningful* musical choices and mathematics struggle to emerge from behind cloaks of terminology, perhaps precariously propped up by constructions of auxiliary geometric objects. The problem is that mathematical content comes in the form of proven statements about well-defined structures, and attempts at "explaining" musical phenomena usually involve structures that are not well defined, with conclusions justified by carefully chosen examples and multitudes of counterexamples ignored. And any logical development of well-defined structure is inevitably based on dubious or pedantic musical principles, so that the resulting conclusions can say precious little about what is important in music.

The types of problems illustrated in the basic examples considered below are compounded in more complicated analytic treatments of music.

Mathematical Explanations of Music

For instance, the recent *Notices* articles [4, 5] use short-time Fourier transforms and continuous wavelet transforms to produce families of images from digital audio and claim to provide insight into musical structure that is both "quantitative" and "objective". The images do exhibit patterns that correspond to rhythmic accents, pitches, and volume, but the analysis of musical content is riddled with flaws and weaknesses that undermine most of the extremely enthusiastic conclusions.

The problems are well illustrated in Example 6 of [4], where four trivial musical observations are made about a short Duke Ellington excerpt: Sometimes symmetries appear in melodies, instruments can bend pitches, jazz can be syncopated, and melodies can contain varying groupings of notes. Areas of the associated images corresponding to these observations are located. It is claimed that "We can see from this analysis that this passage within just six seconds reveals a wealth of structure, including many features that are unique to jazz. Such mastery illustrates why Duke Ellington was one of the greatest composers of the twentieth century." The implication that the examination of the images illustrates anything about the music (let alone the greatness) of Duke Ellington is unfounded for several reasons.

First of all, the "analysis" admittedly includes listening to the recording; the note blobs in the image only contribute frequency readings from one coordinate and indicate rhythmic placement along the time coordinate. The observation of a slurring of pitch together with a brief descending-ascending motif leads the authors to conclude that Ellington is synthesizing "a melodic characteristic of jazz (micro-tones) with one of classical music (reflection about a pitch level)." This conclusion, besides being musically trivial, ignores the fact that symmetries of melodic fragments and bending of pitches (not to mention syncopation) occur in all kinds of music—certainly in both jazz and classical music. The fourth observation refers to a notion of "hierarchy" as giving "preferred" groupings of musical notes via grammar-like rules. But this notion of hierarchy is not well defined, as even recognized in [9] by the authors who coined the notion. And surely the "wealth of structure" visible in the images could also be created by a mediocre or even poor performance of the same or a similar piece. In fact, much richer visual structures could certainly be created by sounds that are more complicated, including sounds that are essentially devoid of musical content. No control examples are given, and the visual data requires listening for interpretation, yet it is claimed "most importantly" that the images "provide an objective description of recorded performances". What does "objective" mean here? Are the authors suggesting that looking at their images provides some true measure of music? Even putting aside the trivial nature of the musical observations, this paragraph makes clear that any meaningful conclusions are in fact being *entirely drawn from listening.*

Example 6 of [5] implies that the images provide an answer to the question: What do Beethoven, Benny Goodman, and Jimi Hendrix have in common? The evidence of "approximate mirror symmetry" is only the trivial observation of melodic lines that descend and then ascend, a property of music that is probably familiar to even the untrained casual listener. Again, all kinds of sounds, including nonmusical ones, could give rise to similar images, and the restricted set of examples contained in [4] and [5] surely reflects the fact that extracting any meaningful general correspondence between the visible patterns and musical content is highly unlikely.

Acclaimed as the first musico-mathematical article to appear in *Science* magazine, [13] claims to illustrate how composers "exploit" the geometry of an orbifold and to show "precisely how harmony and counterpoint are related". Although this article contains well-defined statements and arguments, the weakness of the underlying musical principles erodes any meaningful connection with mathematics. The entire construction is based on the notion of "efficient voice leading", which is justified by the statement that "Western pedagogues instruct composers to minimize voice leading while eschewing crossing changes." In fact, this extremely limited notion can be considered relevant only when it is desired to have an accompaniment that is musically *benign* so as not to interfere with other concurrent statements and is at best a rule of thumb for a student composer/arranger. The experienced creator of music certainly hears every voice and is guided by what sounds best rather than instructions from pedagogues. So, even ignoring some other questionable musical assumptions, it is difficult to derive any conclusions from a geometric construction that is based on a principle that "minimizes" musical content.

Other examples of "geometric" analyses are common, and musical scores written in the time and pitch coordinates of standard notation provide a plethora of patterns and data. The discovery of symmetries and other transformations of musical motifs (as notated) is often presented as evidence of an underlying mathematical component of music. But such discoveries do not correspond to musically coherent or mathematically interesting notions. While repetition and variation pervade music, precise symmetries among musical phrases are certainly not generic, so if such symmetry were musically meaningful, one would expect it to have a recognizable effect. But convincing counterevidence is provided by J. S. Bach's completely palindromic Crab Canon from his Musical Offering. What is remarkable about the Crab Canon is that even the most diligent listener is not going to have a clue that the piece is palindromic without access to the score, and in spite of the extreme notational symmetry the piece sounds characteristically Bach-like and by Bach's standards less memorable than average. (In this case Bach's compositional tour de force is in response to a challenge from Frederick the Great; more on composer-embedded musical patterns will be discussed later.)

A method commonly employed in mathematical analyses of music (including [13]) is to identify pitches that differ by a whole number of octaves, and the resulting equivalence classes are assumed to be a natural object of study. While it is true that pitches that are an octave apart have a clear notion of "sameness" (which is reflected in their shared overtones), the *musical* effect of changing the register of a note (choosing a representative of the pitch class) is not at all negligible. This suggests an interesting experiment: Listen to musical pieces whose pitch class representatives have been randomly permuted. Such shuffling of notes will certainly generate some bizarre-sounding music, and it is a safe bet that your favorite listening would lose its special place in your heart if always subjected to having its notes scattered in this way. But any musical theory that takes seriously the idea of working with pitch classes will apply equally to "explain" such sounds! This modding out by octave "translations" is often invoked by music theorists to construct tori as parameter spaces.

Mathematical Ways of Listening to Music

The second goal of musical analysis raises an interesting question: How does extramusical information affect the listener? The effects are certainly wide ranging, from the relatively benign influences of knowing a song title or anecdotal stories about the performer to the enrapturement of an associated religious ritual. Lyric content or dance generally tends to interact strongly with accompanying musical statements, and when music is presented with video, the music will likely play a subservient role (and in such a setting the power of sound to generate its own images has been compromised). In the case of mathematically oriented music theory, it is usually tacitly assumed that an awareness of any "explanatory" mathematical notions will improve the musical experience. While this may be true for some music theorists, it is important to recognize that it is not necessarily a mathematical insight into essential general musical properties, but more likely a personal enhancement for one who enjoys attaching intellectual constructions to music. In fact, it can often be beneficial to remain ignorant of extramusical information, even when provided by the composer. More than once I have been inspired by music accompanied by lyrics in a language I did not understand only to discover later that the words were not just unrelated to my appreciation but even unappealing to me. More generally, it is remarkable how in spite of the strong link between music and its ambient culture of origin, appreciation of music can bridge wide cultural gaps. For instance, secular appreciation of religious music abounds, the blues can go over well in Asia, hip-hop pieces are sometimes based on loops from classic jazz recordings, and World Music has its own category in the commercial music market. The point here is that, while music comes wrapped in webs of extramusical connections, it is a very subtle matter to extract *essential* threads from the midst of the many personal ones.

The effects of imposing conscious listening techniques often appear in the setting of music pedagogy: The journey from student to professional musician usually involves many years of music theory in the form of organizing sounds into recognizable bits and studying how they interact (there are many methods for doing this). This process of intellectualizing about music is often very difficult, as the student can become hypercritical and overly self-conscious, both as a performer and as a listener. Eventually the experienced musician is able to return to the appreciation of sound for its own sake, retaining the ability to analyze tension and resolution in theoretical terms at will but also free to enjoy the transcendental in-the-moment nature of music.

To clarify, I'm not proposing that analytic listening, mathematically motivated or otherwise, is wrong, just that it is not fundamental to the appreciation of music in general. All kinds of attentive, repeated, and earnest listening can access the full range and depth of musical meaning that is present in sound.

Creation of Music

The most effective use of theory in the creation of music is to provide frameworks for *experimentation* rather than rules to be followed. Again, the methodical organization of sound may motivate the use of mathematical terminology, but while the resulting explorations may help the practicing musician gain insights into subtleties of musical tension and resolution, they are not going to lead to meaningful theorems expressing general essential musical qualities. In fact, even completely arbitrarily formulated methodologies can spark fruitful musical studies (and sometimes give birth to "styles" and "schools") merely by reducing the profusion of available musical choices.

For instance, the various serial composition techniques developed by Western atonal composers such as Schoenberg a century ago involve applications of various formal rules that were designed to avoid traditional combinations of sounds and can be described using elementary mathematical notions like transformations and permutations of pitches and rhythms. But this formalism expressed a self-conscious rebellion against tonality rather than any natural musical structure, and the value of the resulting music always depended, not surprisingly, on the creativity of the composer rather than (or in spite of) the formal structure. By mistaking rigidity (in the colloquial sense) for rigor (in the mathematical sense) such musical formalism is often presented as a "mathematical" aspect of music (e.g., Chapter 8 of [6]). The importance of twentieth-century formalist schools in music has been greatly exaggerated by academics, while the incorporation of dissonance and breaching of harmonic boundaries have proceeded more naturally in the rest of the vast musical world.

While it is not surprising to the mathematician that arbitrary formalism is not mathematics, there is also music that has been created using constructions ostensibly based on mathematical elements (with varying levels of seriousness). However, the inevitable insertion of esthetic choices, together with the arbitrary nature of the underlying constructions, conspires to remove any trace of mathematical content from the picture. For instance, examples of "fractal music" range from simply superimposing melodic fragments over themselves at a few increasing multiples of tempo to multiply iterated computer synthesis of sound from 2-dimensional fractal-like shapes that involves numerous parameter choices. The "poorer approximations" of fractals actually tend to sound more musical, but in any event results certainly do not inspire repeated listening and seem unlikely to produce anything nearly as interesting as properties such as fractional dimension, let alone correspond to any more substantial fractal-related mathematics.

The relationships between the motivations and outputs of artists can be subtle and wide ranging. In the case of mathematically inspired composers it's frequently a matter of "a little knowledge being a dangerous thing", and even for the mathematically astute creator of music there remains the problem of extracting correlation from the inspiration. For instance, when a composer claims that the Fibonacci sequence is essential to one piece of music and then turns around and embeds names into the next piece via rhythmic Morse code, the transient nature of any musico-mathematical relationships is apparent [3]. It is possible to be sincere without being serious, but it is also true that in some circles it can be advantageous for a musician to have a supporting "theory" that critics can latch on to.

Unfortunately, I've yet to hear any mathematically inspired music that comes close to providing the substance and lasting impression of even an elementary piece of reasonably interesting mathematics. This reflects a common occurrence in the art world, where the desire to innovate leads to the celebration of "newness for newness' sake", a phenomenon much less prevalent in mathematics, where the value of new work emerges by consensus rather than by press release and both the audience and the reviewers are mathematicians.

Metaphorical Comparisons

So if the physics of sound is mathematical but not musical and music theory is musical but not mathematical, we can still ask if a common musico-mathematical core is reflected in other, perhaps more metaphorical, ways. Attention will be focused on the question of what might be special to mathematics and music rather than science and art in general.

Fundamental Observation

An interesting web of definitions, theorems, proofs, and conjectures does not require an extramathematical application to be satisfying. Similarly, the rhythmic flow of sonic tensions and resolutions in an instrumental music performance can be appreciated without attributing to the sounds any worldly connotations. In this respect mathematics and music seem to share the property that their content—however subjective and time-dependent—*can* be expressed intrinsically, without direct reference to the natural world of human experience.

Whether you agree or disagree with this statement at face value, I believe it is worth trying to adjust your philosophical viewpoint enough to consider the claim, if only to clarify its limitations. (For instance, if you can't separate any significant part of mathematics or music from the natural world, then at least try to recognize the presence of a significant *degree* of intrinsic meaning.) Since I believe that this observation is important, some clarifications are in order.

First of all, there is clearly an emphasis on "can", because both mathematics and music frequently *do* refer directly to the natural world. While the mathematician is well aware of the subtle and symbiotic interactions between the abstract development of theories and applications of mathematics, analogous interactions also occur with music, which besides being appreciated for its own sake can be associated with lyrics, images, dance, ritual, ceremony, commerce, and other extramusical phenomena. Of course external models are enriching and vital to both disciplines, but it can be helpful to be aware of the distinction, and I believe that the claimed observation of intrinsic meaning provides a special link between mathematics and music.

Among human disciplines this form of intrinsic meaning is essentially *unique* to mathematics and nonlyric music: Other sciences are always directly tied to the natural world via their subject matter, and while other art forms may use abstraction, it almost always involves recognizable elements of human experience that have been distorted or used in unexpected ways.

It is true that certain visual art that is completely devoid of any reference to the natural world can have content, but I feel that the *general* comparison is not even close and that the intrinsic natures of music and mathematics are a significant order of magnitude stronger, although I do not know how to measure this. Some fans of extremely abstract visual art will disagree with me here, and admittedly this may be evidence of a "gray area" where meaning emerges self-referentially from patterns, visual or sonic, perhaps suggesting analogies with certain musical works that seem not to even reference recognizable elements of music. Also relevant here is that the visuals used by mathematicians to express mathematics, such as figures, graphs, and diagrams, can have an esthetic impact of their own, as recognized for instance by the sculpture of Helaman Ferguson (`http://www.helasculpt.com/`). Some might suggest that such images provide more effective artistic embodiments of mathematical ideas than the "pseudorigorous" mathematically inspired music composition techniques discussed above. In any event, I stand by the claim of a significant sense of uniqueness and continue with clarifications.

The locations, characters, and actions in literature and dramatic performance provide essential identifications with the natural world, as even the most fantastic settings inevitably mirror recognizable elements in the lives of the audience. And although the art lies in the development of tension and resolution through changes in relationships among the agents, the effect on the audience is always dependent on qualities and expectations that are inferred from these identifications.

And if the avid poetry listener feels that sometimes the message of the poem is being carried entirely by the cadence, phrasing, texture, and tone of voice of the poet without recognition of any semantic content in the words, then I'd say that what is being heard is *music*. Logical philosophy and computer science can similarly intersect mathematics at their extremes.

Note that this claim of uniqueness is not a denial that other disciplines can have meaning that transcends their inherent references to the natural world, just that what is special to mathematics and music is that their content is capable of being expressed entirely in terms of their own raw material, namely, logical thought and audible sound.

Furthermore, no strict formalist mathematical philosophy is being imposed here, just the acceptance that the contemplation of generalized homology theories, transfinite ordinals, moduli spaces, and the like can (and often must) take place outside the usual realm of sensory perception. We believe that our elements are well defined, that our arguments are satisfyingly checkable, and that mathematics is consistent (although we know we can't prove it). Theories are developed by various internal associations of mathematical elements, but we do not *require* confirmation from an embodiment in human experience; and indeed we don't expect to find such confirmation, since even an object as basic as an interval of real numbers does not have a reliable model in the natural world.

Similarly, no banishment of cultural or other associations with music is being proposed, just the observation that as melodies, rhythms, and harmonies unfold in time, it is the relationships among the sounds that speak to you. The sounds repeat, mutate, diverge, return—always in combination with each other but never in *need* of "pointing" to anything outside the music.

Notice that such frequently recognized qualities as beauty, elegance, power, economy, anticipation, surprise, tension, and resolution are certainly *not* unique to music and mathematics. What is remarkable is that such qualities can emerge at all without need of body language, radiant sunsets, death-defying feats, wireless capabilities, expected rates of return, time travel, or love lost and renewed.

Finally, the claimed uniqueness and extreme level of intrinsic meaning is not intended to imply any judgments on the relative *values* of human endeavors, any of which can of course have a wide range of appeal and utility to a variety of people. In particular, nothing is being implied about the relative importance of "pure" and "applied" in both mathematics and music.

What Do Metaphorical Observations Explain?

The fundamental observation seems to provide a possible reason for the enduring attraction of musico-mathematical investigations: Since the ubiquity and power of mathematical and musical applications are a consequence of the strength of their intrinsic constructions, it is only natural to ask the question, Can they model each other?

But this very modeling power can represent obstructions to an in-depth metaphorical discussion with a general public whose musical and mathematical experiences are dominated by applications. (For instance, instrumental jazz and classical music each account for just a few percent of music sales, which is of course still greater than the publishing share of mathematics journals.) It is an important challenge to somehow share the value of abstract thinking with society at large.

An admirably well-intentioned attempt to describe metaphorical connections between the "inner lives" of music and mathematics to a general audience is the recently reprinted bestseller *Emblems of Mind* [12] by *New York Times* journalist Edward Rothstein. On the positive side, this book brings many worthwhile points to light, including the roles of beauty and creativity in mathematics, the emphasis of relationships over objects, and the power of abstraction inherent in both disciplines. Unfortunately, several fundamental problems cripple the coherent development of the many good ideas present: For instance, the occasionally insightful descriptions of music repeatedly fall into all the traps of musical analysis discussed above. A harbinger of the forthcoming distortion appears in the introduction, where after mentioning musical affinities of Galileo, Euclid, Euler, and Kepler, the author includes Schoenberg, Xenakis, and Cage among a short list of examples that seem to point back from music to mathematics. Even most mathematicians with an affinity for these composers would, with all due respect, surely recognize that this juxtaposition is way out of balance. This leads to such contradictions as claiming the existence of "a systematic logic that guides musical systems", but then admitting later that great musical compositions "create their own form of necessity, the binding coming not from logic but from the unfolding of ideas…". And the spurious metaphorical equating of the contrived formalism of twentieth-century atonal "systems" with the discovery of non-Euclidean geometries both fails to recognize the strong and natural role of modern geometry in mathematics and sidesteps the truth that the natures of tonality and dissonance in music are complicated and mysteriously subtle phenomena that have defied satisfactory explanation by any general theory.

The confusion created by mistaking musical form for content is compounded by being interwoven with an informal poetic analysis of music, frequently laced with fancifully chosen mathematical terminology. While the appreciator of well-written romantic prose may enjoy the exposition, those looking for more substance will be disappointed, as the attempt to nail down details makes the metaphors less robust rather than stronger. For instance, the notion that a "composition proceeds to 'prove' itself" or the claim of an analogue of "completeness" (of a logical system) in music are signs that the discussion is deteriorating. This is confirmed when one of the text's central points relates a metaphorical sense of "truth" in music to musical "style".

One fact clearly underscored by the book is that ordinary human language is much better at conveying mathematical ideas than musical ideas. Although the feeling that music is "telling a story" is often intensely felt by both listener and performer, there is no known well-defined "grammar" of music; and if a picture is worth a thousand words, then the relation between music and language must surely be exponential. On the other hand, mathematics has its set-theoretic foundations expressed in the formal languages of logic, and among mathematicians, informal conversation is the most common method of communicating mathematics. Does this suggest that music is in some sense *more* abstract than mathematics?

The popularity of [12] does confirm that there is a healthily curious audience among the general public. One would hope that such readers could be encouraged to pursue their investigation of mathematics in the growing number of expository sources written by mathematicians, such as the recent *Princeton Companion to Mathematics* [7] (although the brief section on mathematics and music in [7] gives too much weight to the type of superficial musical analysis criticized above).

Creative Processes

One might summarize the essence of a very general metaphorical view by the statement that "mathematics and music are the science and art of analogy". Although it appears to be difficult to extract more precision from metaphors, I believe that, by focusing on mathematical and musical *creative processes*, useful conclusions can be drawn.

In fact, the process of creating or discovering mathematics is in many ways analogous to a small-group jazz performance: This is evident in the real-time exchange of ideas among collaborators, spontaneously alternating lead and accompaniment roles, guided by a thematic problem, developing material statement by statement, pursuing tangential ideas, adapting to mistakes, being ready for unexpected results, and never knowing for sure if the original goals will be achieved. I believe that this analogy with musical improvisation is stronger than any picture of the mathematician as the solitary composer (although the most vital composers do capture the spirit of improvisation in their works), as there is a sense in which the nonperforming composer can rework the landscape to "force his theorems to be true" (but not necessarily "interesting"), whereas the improvisor must face the unforgiving judgment of the moment while traveling without a seatbelt. The analogy also extends to the researcher working alone as a solo improvisor, simultaneously playing lead and accompaniment roles as the devil's advocate, and even to the processes of understanding mathematics and interpreting composed music. (Note that the tradition of improvisation in Western classical music, which stretches back through Beethoven, Mozart, and Bach, shows signs of a rebirth [11].)

But this improvisational analogy can apply more generally to processes involved in many human endeavors, not only in the arts and sciences but also including many workplace environments encountered by citizens of today's fast-changing global society. In fact, in the face of turbulent economic conditions, advancing technologies, and increasingly international markets, employers and employees alike are going to be dealing with shifting work flows and new job types and products, as well as interactions with foreign cultures, all of which will require creative problem solving to recognize appropriate skill sets, implement effective training and study methods, and develop new career and employment programs.

The key point here is that the intrinsic nature of mathematics and music suggests that the studies of both research mathematics and improvisational music could play valuable roles in modern education, as their *abstract yet cohesive structures serve as models* for developing flexible skills and the ability to generate spontaneous constructive thought. While the problem-solving techniques and computational powers of mathematics are already well appreciated, the more abstract, creative, and improvisational aspects of human thought are going to be increasingly valuable in twenty-first-century life.

Ideally these studies would be completely integrated into the education system, with the associated musical and mathematical learning processes naturally complementing and reinforcing each other. What is important here is that the *goals of research and improvisation guide the pedagogy.* The challenge is to develop courses, programs, and teaching conceptions with these goals in mind and to incorporate them into the curriculum. (Note that combining mathematics and music in the classroom is not being proposed here.)

That the underlying frameworks of the studies can complement and reinforce each other is apparent at many levels. For instance, the student of musical improvisation uses formalism (music theory) to generate examples (sounds) that are examined esthetically (by listening), while the student of mathematics generates examples (special cases) to understand formalism (general statements) that are considered logically (by proving/disproving). More generally, both studies develop experience with solitary practice, group work, and open-ended learning. Many other such pedagogical frameworks exist at all levels and age groups.

The idea is not to produce more professional mathematicians and musicians (although talent would be more likely to flourish), but rather to provide greater general access and exposure to the relevant abstract skills. Of course some will benefit more from musical study, others from mathematics, and both subjects will still be challenging for almost everyone. But the recognition of the long-term benefits should provide motivation, and effective integration into the education structure would provide support to maximize the positive value for as many as possible. *The almost complete ignorance of the essences of mathematical research and improvisational music that is prevalent in society today means that the initial marginal benefits could be enormous.*

Of course the challenges faced in implementing such an educational vision would be huge, as effective teaching of both research mathematics and improvisational music is already difficult enough, and the skeptic will point to the existing body of inconclusive studies regarding musical and mathematical pedagogical methodology, as well as the apparent lack of supporting circumstantial evidence (where are all the improvisational music groups of mathematical researchers?). But there are good reasons to believe that the obstacles are surmountable and that the vision is valid: First of all, there is a growing consensus supporting educational reform, as well as funding available for innovative ideas. The mathematical research community has shown purposeful commitment to teaching in recent years, while at the same time the many jazz departments in universities and colleges across the country have become increasingly populated with top-level faculty having significant performance experience. (So a pilot program for preparing teachers could involve cross-training of graduate and/or undergraduate students, for example.) And although music has been largely cut from primary and secondary school curricula, the many independent organizations that have been providing music instruction could provide infrastructure for pilot programs on the musical side. On the mathematical side, a new vision is desperately needed to guide a complete reforming of the current generally dreadful state of mathematics education at the primary and secondary school levels. That aspects of this vision have already been accepted is evidenced by the increasing numbers of mathematics Ph.D.s working outside academia [10] and by the direct implementation of jazz conceptions in high-level business consulting [8].

Existing educational data should not be expected to provide insight into the worth of the proposed vision, primarily since such a focus on research and improvisation has not been significantly implemented. I would also expect that direct effects will be difficult to measure, especially in the short term. The problem of correlating success in varying job types is in itself an interesting problem in today's ocean of information and shifting employment patterns. And although I know of various successful external applications of musical and mathematical frames of mind, the satisfying nature of improvisational and research experiences means that those who are good at it are likely to happily stay with it.

Conclusion

It is clear that an in-depth appreciation of both mathematics and music is a prerequisite to the critical consideration of musico-mathematical relationships and their kernels. But to the extent that one appreciates mathematics, one *is* a mathematician, whereas the appreciator of music need not be a musician. It follows that the mathematics community is likely to provide constructive contributors to the dialogue. Expositing and teaching mathematics (independently of music), as well as promoting exposure to all forms of music, will contribute to opening the discussion to a wider audience. Ideally this could be integrated into the entire educational system. At least, one would hope that inviting metaphors might provide motivation for deeper exploration and in particular lead to a wider awareness of the esthetics of mathematics. The danger is that the unconscious readiness with which the mind accepts analogies will allow poetic hand waving to stir up pleasing but shallow illusions, clouding a picture that can only be clarified by thoughtful hard work.

In an ideal world, a marriage of mathematics and music should celebrate the beauty and power of abstraction. But the courtship is thrown off balance by the contrast between the open-access nature of the musical world, in which the listener is free to navigate by ear, and the rigor of the mathematical world, in which the curious mind must temper its imagination with logic. The proliferation of suitors in the natural world further complicates matters, rendering detailed agreements, scientific or metaphorical, elusive. In spite of the voluminous literature inspired by this undeniably intriguing situation, many of the most salient observations on the subject are one-liners, often provided by mathematicians (see e.g., [1]). On that note, I would like to provide an affirmative answer to the title question by offering a punch line of my own: "Mathematics is like music that only musicians can hear."

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